HW #8 Homomorphisms - Solutions

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In Exercises 1-9, determine whether or not the given function defines a homomorphism. If ϕ IS an isomorphism, identify $\ker(\phi)$.

1. $\phi : (\mathbb{Z}, +) \to (\mathbb{R}, +)$, given by $\phi(n) = n$

   The operation "+" is the same operation in $\mathbb{R}$ as it is in $\mathbb{Z}$.

   Consequently, $\phi(m+n) = m+n = \phi(m) + \phi(n)$, $\forall\ m, n \in \mathbb{Z}$

   (i.e., $\phi(m+n) = \phi(m) + \phi(n)$, $\forall\ m, n \in \mathbb{Z}$.)

   Hence, ϕ is a homorphism.

   Recall that $\ker(\phi) = \{n \in \mathbb{Z} : \phi(n) = 0\}$

   Note that since $\phi(n) = n$, then $\phi(n) = 0$ implies that $n = 0$

   Thus, $\ker(\phi) = \{0\}$

ϕ IS a homorphism. $\ker(\phi) = \{0\}$

2. $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$, given by $\phi(n) = -n$

   Note that $\phi(m+n) = -(m+n) = -m-n = (-m) + (-n) = \phi(m) + \phi(n)$, $\forall\ m, n \in \mathbb{Z}$

   (i.e., $\phi(m+n) = \phi(m) + \phi(n)$, $\forall\ m, n \in \mathbb{Z}$.)

   Hence, ϕ is an homorphism.

   Recall that $\ker(\phi) = \{n \in \mathbb{Z} : \phi(n) = 0\}$

   Note that since $\phi(n) = -n$, then $\phi(n) = 0$ implies that $-n = 0$, which implies that $n = 0$.

   Thus, $\ker(\phi) = \{0\}$

ϕ IS a homorphism. $\ker(\phi) = \{0\}$

3. $\phi : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{R} \setminus \{0\}, \cdot)$, given by $\phi(x) = |x|$

   Observe that $\phi(x \cdot y) = \underbrace{|x \cdot y| = |x| \cdot |y|}_{|xy|=|x||y| \ \forall x,y \in \mathbb{R}} = \phi(x) \cdot \phi(y), \quad \forall\, x, y \in \mathbb{R} \setminus \{0\}$

   (i.e., $\phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad \forall\, x, y \in \mathbb{R} \setminus \{0\}$.)

   Hence, ϕ is a homorphism.

   Recall that $\ker(\phi) = \{x \in \mathbb{R} \setminus \{0\} : \phi(x) = 1\}$

   Note that since $\phi(x) = |x|$, then $\phi(x) = 1$ implies that $|x| = 1$, which implies that $x = 1$.

   Thus, $\ker(\phi) = \{1\}$

   $\boxed{\phi \text{ IS a homorphism. } \ker(\phi) = \{1\}}$

4. $\phi : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{R} \setminus \{0\}, \cdot)$, given by $\phi(x) = \frac{1}{x}$

   Observe that $\phi(x \cdot y) = \frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y} = \phi(x) \cdot \phi(y), \quad \forall\, x, y \in \mathbb{R} \setminus \{0\}$

   (i.e., $\phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad \forall\, x, y \in \mathbb{R} \setminus \{0\}$.)

   Hence, ϕ is a homorphism.

   Recall that $\ker(\phi) = \{x \in \mathbb{R} \setminus \{0\} : \phi(x) = 1\}$

   Note that since $\phi(x) = x$, then $\phi(x) = 1$ implies that $x = 1$

   Thus, $\ker(\phi) = \{1\}$

   $\boxed{\phi \text{ IS a homorphism. } \ker(\phi) = \{1\}}$

5. $\phi : (\mathbb{Z}_6, +) \to (\mathbb{Z}_2, +)$, given by $\phi(n) = n \bmod 2$

   Observe that $\phi(m + n) = (m + n) \bmod 2 = m \bmod 2 + n \bmod 2 = \phi(m) + \phi(n), \quad \forall\, m, n \in \mathbb{Z}_6$

   (i.e., $\phi(m + n) = \phi(m) + \phi(n), \quad \forall\, m, n \in \mathbb{Z}_6$.)

   Hence, ϕ is a homorphism.

   Recall that $\ker(\phi) = \{n \in \mathbb{Z} : \phi(n) = 0\}$

   Note that $\phi(n) = 0$, for $n = 0, 2, 4$.

   Thus, $\ker(\phi) = \{0, 2, 4\}$

   $\boxed{\phi \text{ IS a homorphism. } \ker(\phi) = \{0, 2, 4\}}$

6. $\phi : (\mathbb{Z}_9, +) \to (\mathbb{Z}_2, +)$, given by $\phi(x) = n \bmod 2$

   Observe: $\phi(7+7) = \phi(5) = 1$

   $\phi(7) + \phi(7) = 1 + 1 = 0$

   i.e. $\phi(7+7) \neq \phi(7) + \phi(7)$.

   Alternatively: Since $(\mathbb{Z}_9, +)$ and $(\phi[\mathbb{Z}_9], +) = (\mathbb{Z}_2, +)$ are both cyclic, ϕ would have to map generator to generator, in order for ϕ to be a homomorphism. But note that 8 is a generator of $(\mathbb{Z}_9, +)$. (This is easier to see if we note that 8 is the inverse of 1 in $(\mathbb{Z}_9, +)$, and hence, 8 is a generator $(\mathbb{Z}_9, +)$ also.)

   Back to our point: ϕ would have to map generator to generator, in order for ϕ to be a homomorphism. But ϕ maps 8, which is a generator of $(\mathbb{Z}_9, +)$, to 0, which does NOT generate $(\phi[\mathbb{Z}_9], +) = (\mathbb{Z}_2, +)$.
   Hence, ϕ is NOT an homorphism.

Hence, ϕ is NOT a homorphism.

7. $\phi : (\mathbb{R}, +) \to (\mathbb{R} \times \mathbb{R}, +)$, given by $\phi(x) = (x, 0)$

   Observe that $\phi(x+y) = (x+y, 0) = (x, 0) + (y, 0) = \phi(x) + \phi(y), \quad \forall\, x, y \in \mathbb{R}$

   (i.e., $\phi(x+y) = \phi(x) + \phi(y), \quad \forall\, x, y \in \mathbb{R}$.)

   Hence, ϕ is a homorphism.

   Recall that $\ker(\phi) = \{x \in \mathbb{R} : \phi(x) = (0, 0)\}$

   Note that since $\phi(x) = (x, 0)$, then $\phi(x) = (0, 0)$ implies that $x = 0$

   Thus, $\ker(\phi) = \{0\}$

ϕ IS a homorphism. $\ker(\phi) = \{0\}$

8. $\phi : (\mathbb{Z}_5, +) \to (\mathbb{Z}_{25}, +)$, given by $\phi(x) = x \bmod 25$

   Observe that $\phi(2+3) = \phi(0) = 0 \bmod 25 \neq 2 \bmod 25 + 3 \bmod 25 = \phi(2) + \phi(3)$.

   (i.e., $\phi(2+3) \neq \phi(2) + \phi(3)$.)

Hence, ϕ is NOT a homorphism.

9. $\phi : (\mathbb{Z}_5, +) \to (\mathbb{Z}_{27}, +)$, given by $\phi(x) = x \bmod 27$

Note (by the way that ϕ is defined) that if ϕ IS a homomorphism, then $\ker(\phi) = \{0\}$.

(i.e., Since we know that $\phi(1) = 1 \bmod 27$, and $\phi(2) = 2 \bmod 27$, etc., we also know that $\ker(\phi) = \{0\}$.)

Thus, ϕ is one to one.

Note also that ϕ is onto $\phi[\mathbb{Z}_5]$

Thus, $\phi(\mathbb{Z}_5, +) \to (\phi[\mathbb{Z}_5], +)$ should be an isomorphism

Therefore, $(\phi[\mathbb{Z}_5], +)$ should be a group of order 5

But then this would make $(\phi[\mathbb{Z}_5], +)$ a subgroup of $(\mathbb{Z}_{27}, +)$.

But this is impossible, since the **order** of a subgroup $(\phi[\mathbb{Z}_5], +)$ (which is 5) must divide the order of the group $(\mathbb{Z}_{27}, +)$.

So ϕ is NOT a homomorphism.

Alternatively: Observe that $\phi(2+3) = \phi(0) = 0 \bmod 27 \neq 2 \bmod 27 + 3 \bmod 27 = \phi(2) + \phi(3)$.

Hence, ϕ is NOT a homorphism.