# MTH 4441 HW #9 - Cosets - Solutions

FALL 2017

Pat Rossi Name ______________________

1. ~

(a) $(4\mathbb{Z},+)$ is a subgroup of $(\mathbb{Z},+)$. Find all of the cosets of $4\mathbb{Z}$.

Note that $(\mathbb{Z},+)$ is abelian. Therefore every subgroup is a normal subgroup. This means that every left coset is a right coset and vice versa.

Consequently, we will compute the left cosets of the form: $(a+4\mathbb{Z})$, and let $(a+4\mathbb{Z})$ represent both the left and right coset.

**Observe:**

$0+4\mathbb{Z}=\{\ldots,0+(-12),0+(-8),0+(-4),0+0,0+4,0+8,0+12,\ldots\}$

$=\{\ldots,-12,-8,-4,0,4,8,12,\ldots\}=4\mathbb{Z}$

i.e., $0+4\mathbb{Z}=4\mathbb{Z}=\{\ldots,-12,-8,-4,0,4,8,12,\ldots\}$

$1+4\mathbb{Z}=\{\ldots,1+(-12),1+(-8),1+(-4),1+0,1+4,1+8,1+12,\ldots\}$

$=\{\ldots,-11,-7,-3,1,5,9,13,\ldots\}$

$2+4\mathbb{Z}=\{\ldots,2+(-12),2+(-8),2+(-4),2+0,2+4,2+8,2+12,\ldots\}$

$=\{\ldots,-10,-6,-2,2,6,10,14,\ldots\}$

$3+4\mathbb{Z}=\{\ldots,3+(-12),3+(-8),3+(-4),3+0,3+4,3+8,3+12,\ldots\}$

$=\{\ldots,-9,-5,-1,3,7,11,15,\ldots\}$

**Next Observe:** The cosets $4\mathbb{Z},1+4\mathbb{Z},2+4\mathbb{Z},3+4\mathbb{Z}$ partition $\mathbb{Z}$ into sets that are [1]mutually exclusive and [2]collectively exhaustive. (i.e., disjoint sets whose union is the entire set $\mathbb{Z}$.)

Therefore, the cosets $4\mathbb{Z},1+4\mathbb{Z},2+4\mathbb{Z},3+4\mathbb{Z}$ are ALL of the cosets of $4\mathbb{Z}$. There are no more.

(b) Create a group table for the Factor Group $(\mathbb{Z}/4\mathbb{Z},+)$. (i.e., the cosets of $4\mathbb{Z}$ form a group under the operation of "coset addition." Create a group table for the cosets of $4\mathbb{Z}$.)

**Remark:** addition of cosets $(a+4\mathbb{Z})$ and $(b+4\mathbb{Z})$ is computed as follows:

$$(a+4\mathbb{Z})+(b+4\mathbb{Z})=((a+b)+4\mathbb{Z})$$

For example: $(1+4\mathbb{Z})+(2+4\mathbb{Z})=((1+2)+4\mathbb{Z})=(3+4\mathbb{Z})$

Also: $(2+4\mathbb{Z})+(3+4\mathbb{Z})=((2+3)+4\mathbb{Z})=(5+4\mathbb{Z})=(1+4\mathbb{Z})$

The Group Table for the Factor Group $(\mathbb{Z}/4\mathbb{Z},+)$ is:

| $+$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ |
|---|---|---|---|---|
| $4\mathbb{Z}$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ |
| $(1+4\mathbb{Z})$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ |
| $(2+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ |
| $(3+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ |

2. ~

(a) $(4\mathbb{Z}, +)$ is a subgroup of $(2\mathbb{Z}, +)$. Find all of the cosets of $4\mathbb{Z}$.

Note that $(2\mathbb{Z}, +)$ is abelian. Therefore every subgroup is a normal subgroup. This means that every left coset is a right coset and vice versa.

Consequently, we will compute the left cosets of the form: $(a + 4\mathbb{Z})$, and let $(a + 4\mathbb{Z})$ represent both the left and right coset.

**Observe:**

$$0 + 4\mathbb{Z} = \{\ldots, 0 + (-12), 0 + (-8), 0 + (-4), 0 + 0, 0 + 4, 0 + 8, 0 + 12, \ldots\}$$

$$= \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\} = 4\mathbb{Z}$$

i.e., $0 + 4\mathbb{Z} = 4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$

$$2 + 4\mathbb{Z} = \{\ldots, 2 + (-12), 2 + (-8), 2 + (-4), 2 + 0, 2 + 4, 2 + 8, 2 + 12, \ldots\}$$

$$= \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\}$$

**Next Observe:** The cosets $4\mathbb{Z}, 2 + 4\mathbb{Z}$, partition $2\mathbb{Z}$ into sets that are [1]mutually exclusive and [2]collectively exhaustive. (i.e., disjoint sets whose union is the entire set $2\mathbb{Z}$.)

Therefore, the cosets $4\mathbb{Z}, 2 + 4\mathbb{Z}$, are ALL of the cosets of $4\mathbb{Z}$. There are no more.

(b) Create a group table for the Factor Group $(2\mathbb{Z}/4\mathbb{Z}, +)$. (i.e., the cosets of $4\mathbb{Z}$ form a group under the operation of "coset addition." Create a group table for the cosets of $4\mathbb{Z}$.)

**Remark:** addition of cosets $(a + 4\mathbb{Z})$ and $(b + 4\mathbb{Z})$ is computed as follows:

$$(a + 4\mathbb{Z}) + (b + 4\mathbb{Z}) = ((a + b) + 4\mathbb{Z})$$

For example: $(2 + 4\mathbb{Z}) + (2 + 4\mathbb{Z}) = ((2 + 2) + 4\mathbb{Z}) = (4 + 4\mathbb{Z}) = (0 + 4\mathbb{Z}) = 4\mathbb{Z}$

| $+$ | $4\mathbb{Z}$ | $(2 + 4\mathbb{Z})$ |
|---|---|---|
| $4\mathbb{Z}$ | $4\mathbb{Z}$ | $(2 + 4\mathbb{Z})$ |
| $(2 + 4\mathbb{Z})$ | $(2 + 4\mathbb{Z})$ | $4\mathbb{Z}$ |

3. ~

(a) $\langle 3 \rangle$ is a cyclic subgroup of $(\mathbb{Z}_{12}, +)$. Find all of the cosets of $\langle 3 \rangle$.

Recall:

$3 + 3 = 6$

$3 + 2(3) = 9$

$3 + 3(3) = 0$

$3 + 4(3) = 3$

i.e., $\langle 3 \rangle = \{0, 3, 6, 9\}$

Note that $(\mathbb{Z}_{12}, +)$ is abelian. Therefore every subgroup is a normal subgroup. This means that every left coset is a right coset and vice versa.

Consequently, we will compute the left cosets of the form: $(a + \langle 3 \rangle)$, and let $(a + \langle 3 \rangle)$ represent both the left and right coset.

**Observe:**

$0 + \langle 3 \rangle = \{0 + 0, 0 + 3, 0 + 6, 0 + 9\} = \{0, 3, 6, 9\} = \langle 3 \rangle$

i.e., $0 + \langle 3 \rangle = \langle 3 \rangle = \{0, 3, 6, 9\}$

$1 + \langle 3 \rangle = \{1 + 0, 1 + 3, 1 + 6, 1 + 9\} = \{1, 4, 7, 10\}$

$2 + \langle 3 \rangle = \{2 + 0, 2 + 3, 2 + 6, 2 + 9\} = \{2, 5, 8, 11\}$

**Next Observe:** The cosets $\langle 3 \rangle, 1 + \langle 3 \rangle, 2 + \langle 3 \rangle$ partition $\mathbb{Z}_{12}$ into sets that are [1]mutually exclusive and [2]collectively exhaustive. (i.e., disjoint sets whose union is the entire set $\mathbb{Z}_{12}$.)

Therefore, the cosets $\langle 3 \rangle, 1 + \langle 3 \rangle, 2 + \langle 3 \rangle$ are ALL of the cosets of $\langle 3 \rangle$. There are no more.

(b) Create a group table for the Factor Group $(\mathbb{Z}_{12}/\langle 3 \rangle, +)$. (i.e., the cosets of $\langle 3 \rangle$ form a group under the operation of "coset addition." Create a group table for the cosets of $\langle 3 \rangle$.)

**Remark:** addition of cosets $(a + \langle 3 \rangle)$ and $(b + \langle 3 \rangle)$ is computed as follows:

$$(a + \langle 3 \rangle) + (b + \langle 3 \rangle) = ((a + b) + \langle 3 \rangle)$$

For example: $(1 + \langle 3 \rangle) + (2 + \langle 3 \rangle) = ((1 + 2) + \langle 3 \rangle) = (3 + \langle 3 \rangle) = \langle 3 \rangle$

The Group Table for the Factor Group $(\mathbb{Z}_{12}/\langle 3 \rangle, +)$ is:

| $+$ | $\langle 3 \rangle$ | $(1 + \langle 3 \rangle)$ | $(2 + \langle 3 \rangle)$ |
|---|---|---|---|
| $\langle 3 \rangle$ | $\langle 3 \rangle$ | $(1 + \langle 3 \rangle)$ | $(2 + \langle 3 \rangle)$ |
| $(1 + \langle 3 \rangle)$ | $(1 + \langle 3 \rangle)$ | $(2 + \langle 3 \rangle)$ | $\langle 3 \rangle$ |
| $(2 + \langle 3 \rangle)$ | $(2 + \langle 3 \rangle)$ | $\langle 3 \rangle$ | $(1 + \langle 3 \rangle)$ |

4. ~

(a) $\langle 4 \rangle$ is a cyclic subgroup of $(\mathbb{Z}_{12}, +)$. Find all of the cosets of $\langle 4 \rangle$.

Recall:

$4 + 4 = 8$

$4 + 2(4) = 0$

$4 + 3(4) = 4$

i.e., $\langle 4 \rangle = \{0, 4, 8\}$

Note that $(\mathbb{Z}_{12}, +)$ is abelian. Therefore every subgroup is a normal subgroup. This means that every left coset is a right coset and vice versa.

Consequently, we will compute the left cosets of the form: $(a + \langle 4 \rangle)$, and let $(a + \langle 4 \rangle)$ represent both the left and right coset.

**Observe:**

$0 + \langle 4 \rangle = \{0 + 0, 0 + 4, 0 + 8\} = \{0, 4, 8\} = \langle 4 \rangle$

i.e., $0 + \langle 4 \rangle = \langle 4 \rangle = \{0, 4, 8\}$

$1 + \langle 4 \rangle = \{1 + 0, 1 + 4, 1 + 8\} = \{1, 5, 9\}$

$2 + \langle 4 \rangle = \{2 + 0, 2 + 4, 2 + 8\} = \{2, 6, 10\}$

$3 + \langle 4 \rangle = \{3 + 0, 3 + 4, 3 + 8\} = \{3, 7, 11\}$

**Next Observe:** The cosets $\langle 4 \rangle, 1 + \langle 4 \rangle, 2 + \langle 4 \rangle, 3 + \langle 4 \rangle$ partition $\mathbb{Z}_{12}$ into sets that are [1]mutually exclusive and [2]collectively exhaustive. (i.e., disjoint sets whose union is the entire set $\mathbb{Z}_{12}$.)

Therefore, the cosets $\langle 4 \rangle, 1 + \langle 4 \rangle, 2 + \langle 4 \rangle$, and $3 + \langle 4 \rangle$ are ALL of the cosets of $\langle 4 \rangle$. There are no more.

(b) Create a group table for the Factor Group $(\mathbb{Z}_{12}/\langle 4 \rangle, +)$. (i.e., the cosets of $\langle 4 \rangle$ form a group under the operation of "coset addition." Create a group table for the cosets of $\langle 4 \rangle$.)

**Remark:** addition of cosets $(a + \langle 4 \rangle)$ and $(b + \langle 4 \rangle)$ is computed as follows:

$$(a + \langle 4 \rangle) + (b + \langle 4 \rangle) = ((a + b) + \langle 4 \rangle)$$

For example: $(1 + \langle 4 \rangle) + (2 + \langle 4 \rangle) = ((1 + 2) + \langle 4 \rangle) = (3 + \langle 4 \rangle)$

The Group Table for the Factor Group $(\mathbb{Z}_{12}/\langle 3 \rangle, +)$ is:

| $+$ | $\langle 4 \rangle$ | $(1 + \langle 4 \rangle)$ | $(2 + \langle 4 \rangle)$ | $(3 + \langle 4 \rangle)$ |
|---|---|---|---|---|
| $\langle 4 \rangle$ | $\langle 4 \rangle$ | $(1 + \langle 4 \rangle)$ | $(2 + \langle 4 \rangle)$ | $(3 + \langle 4 \rangle)$ |
| $(1 + \langle 4 \rangle)$ | $(1 + \langle 4 \rangle)$ | $(2 + \langle 4 \rangle)$ | $(3 + \langle 4 \rangle)$ | $\langle 4 \rangle$ |
| $(2 + \langle 4 \rangle)$ | $(2 + \langle 4 \rangle)$ | $(3 + \langle 4 \rangle)$ | $\langle 4 \rangle$ | $(1 + \langle 4 \rangle)$ |
| $(3 + \langle 4 \rangle)$ | $(3 + \langle 4 \rangle)$ | $\langle 4 \rangle$ | $(1 + \langle 4 \rangle)$ | $(2 + \langle 4 \rangle)$ |

5. ~

(a) Show that $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$, given by $\phi(x) = x \bmod 4$, is a homomorphism. (The operation in $(\mathbb{Z}_4, +)$ is addition modulo 4.)

**Recall:** $\forall x, y \in \mathbb{Z}$, and $\forall$ natural numbers $n \geq 2$ :

$\phi(x+y) = (x+y) \bmod n = [(x \bmod n) + (y \bmod n)] \bmod n = \phi(x) + \phi(y)$

Therefore, $\forall x, y \in \mathbb{Z}$, $\quad \phi(x+y) = (x+y) \bmod 4 = [(x \bmod 4) + (y \bmod 4)] \bmod 4 = \phi(x) + \phi(y)$

(b) Identify $\ker(\phi)$, and compute the left and right cosets of $\ker(\phi)$.

By definition of $\ker(\phi)$, $\ker(\phi) = \{x \in \mathbb{Z} : \phi(x) = 0\}$

Since $\phi(x) = x \bmod 4$, $\quad \ker(\phi) = \{x \in \mathbb{Z} : x = 0 \bmod 4\} = \{x \in \mathbb{Z} : x = q(4) + 0 \text{ for } q \in \mathbb{Z}\}$

i.e., $\ker(\phi) = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\} = 4\mathbb{Z}$

The kernel of a homomorphism is always a normal subgroup of the original group. This means that every left coset of $\ker(\phi)$ is also a right coset of $\ker(\phi)$, and vice versa.

Consequently, we will compute the left cosets of the form: $(a + 4\mathbb{Z})$, and let $(a + 4\mathbb{Z})$ represent both the left and right coset.

To find all of the cosets of $\ker(\phi) = 4\mathbb{Z}$, observe that:

$0 + 4\mathbb{Z} = 0 + \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$

$\qquad = \{\ldots, 0 + (-12), 0 + (-8), 0 + (-4), 0 + 0, 0 + 4, 0 + 8, 0 + 12, \ldots\}$

$\qquad = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\} = 4\mathbb{Z}$

i.e., $0 + 4\mathbb{Z} = 4\mathbb{Z}$

$1 + 4\mathbb{Z} = 1 + \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$

$\qquad = \{\ldots, 1 + (-12), 1 + (-8), 1 + (-4), 1 + 0, 1 + 4, 1 + 8, 1 + 12, \ldots\}$

$\qquad = \{\ldots, -11, -7, -3, 0, 5, 9, 13, \ldots\}$

i.e., $1 + 4\mathbb{Z} = \{\ldots, -11, -7, -3, 0, 5, 9, 13, \ldots\}$

$2 + 4\mathbb{Z} = 2 + \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$

$\qquad = \{\ldots, 2 + (-12), 2 + (-8), 2 + (-4), 2 + 0, 2 + 4, 2 + 8, 2 + 12, \ldots\}$

$\qquad = \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\}$

i.e., $2 + 4\mathbb{Z} = \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\}$

$3 + 4\mathbb{Z} = 3 + \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$

$\qquad = \{\ldots, 3 + (-12), 3 + (-8), 3 + (-4), 3 + 0, 3 + 4, 3 + 8, 3 + 12, \ldots\}$

$\qquad = \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}$

i.e., $3 + 4\mathbb{Z} = \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}$

**Next Observe:** The cosets $4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}$ partition $\mathbb{Z}$ into sets that are [1]mutually exclusive and [2]collectively exhaustive. (i.e., disjoint sets whose union is the entire set $\mathbb{Z}$.)

Therefore, the cosets $4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}$ are ALL of the cosets of $\mathbb{Z}$. There are no more.

(c) Create a group table for the Factor Group $(\mathbb{Z}/\ker(\phi), +)$. (i.e., the cosets of $\ker(\phi)$ form a group under the operation of "coset addition." Create a group table for the cosets of $\ker(\phi)$.)

**Remark:** addition of cosets $(a+4\mathbb{Z})$ and $(b+4\mathbb{Z})$ is computed as follows:

$$(a+4\mathbb{Z})+(b+4\mathbb{Z})=(a+b)+4\mathbb{Z}$$

For example: $(1+4\mathbb{Z})+(2+4\mathbb{Z})=((1+2)+4\mathbb{Z})=(3+4\mathbb{Z})$

Another example: $(2+4\mathbb{Z})+(3+4\mathbb{Z})=((2+3)+4\mathbb{Z})=(5+4\mathbb{Z})=(1+4\mathbb{Z})$

Notice that in general, $(a+4\mathbb{Z})+(b+4\mathbb{Z})=(a+b) \bmod 4+4\mathbb{Z}$

The Group Table for the Factor Group $(\mathbb{Z}/\ker(\phi), +)=(\mathbb{Z}/4\mathbb{Z}, +)$ is:

| $+$ | $\ker(\phi)$ | $(1+\ker(\phi))$ | $(2+\ker(\phi))$ | $(3+\ker(\phi))$ |
|---|---|---|---|---|
| $\ker(\phi)$ | $\ker(\phi)$ | $(1+\ker(\phi))$ | $(2+\ker(\phi))$ | $(3+\ker(\phi))$ |
| $(1+\ker(\phi))$ | $(1+\ker(\phi))$ | $(2+\ker(\phi))$ | $(3+\ker(\phi))$ | $\ker(\phi)$ |
| $(2+\ker(\phi))$ | $(2+\ker(\phi))$ | $(3+\ker(\phi))$ | $\ker(\phi)$ | $(1+\ker(\phi))$ |
| $(3+\ker(\phi))$ | $(3+\ker(\phi))$ | $\ker(\phi)$ | $(1+\ker(\phi))$ | $(2+\ker(\phi))$ |

In terms of $4\mathbb{Z}$, this becomes:

| $+$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ |
|---|---|---|---|---|
| $4\mathbb{Z}$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ |
| $(1+4\mathbb{Z})$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ |
| $(2+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ |
| $(3+4\mathbb{Z})$ | $(3+4\mathbb{Z})$ | $4\mathbb{Z}$ | $(1+4\mathbb{Z})$ | $(2+4\mathbb{Z})$ |

(d) Define an isomorphism $\mu : (\mathbb{Z}/\ker(\phi), +) \to (\mathbb{Z}_4, +)$ same as $\mu : (\mathbb{Z}/4\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$

The map that we seek is $\mu(a+4\mathbb{Z})=a \bmod 4$

Observe (from an observation in 5.c) that for cosets $(a+4\mathbb{Z})$ and $(b+4\mathbb{Z})$:

$(a+4\mathbb{Z})+(b+4\mathbb{Z})=(a+b) \bmod 4+4\mathbb{Z}$

Thus, $\mu[(a+4\mathbb{Z})+(b+4\mathbb{Z})]=\mu[(a+b) \bmod 4+4\mathbb{Z}]=\mu[(a \bmod 4+b \bmod 4)+4\mathbb{Z}]=a \bmod 4+ b \bmod 4=\mu(a+4\mathbb{Z})+\mu(b+4\mathbb{Z})$

i.e., $\mu[(a+4\mathbb{Z})+(b+4\mathbb{Z})]=\mu(a+4\mathbb{Z})+\mu(b+4\mathbb{Z})$

**Furthermore:** $\mu(4\mathbb{Z})=0$
$\mu(1+4\mathbb{Z})=1$
$\mu(2+4\mathbb{Z})=2$
$\mu(3+4\mathbb{Z})=3$

i.e., $\mu : (\mathbb{Z}/4\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$ is one to one and onto.

Hence, $\mu : (\mathbb{Z}/4\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$ given by $\mu(a+4\mathbb{Z})=a \bmod 4$ is an isomorphism