MTH 4441 - Test 2 - Solutions

FALL 2017

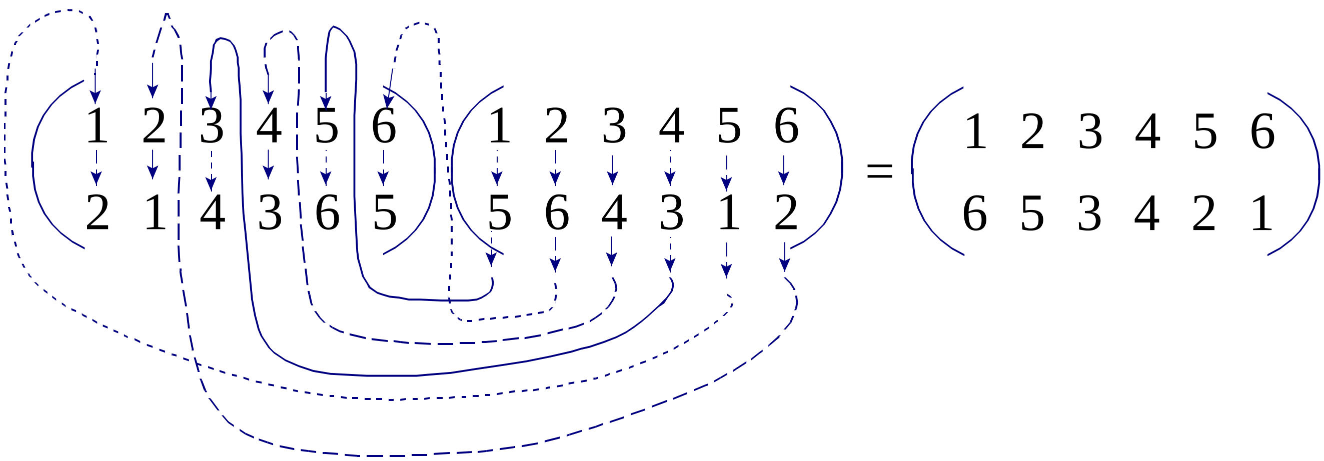
Pat Rossi Name ______________________

Show CLEARLY how you arrive at your answers.

In Exercises 1-4, compute the "product" of the permutations.

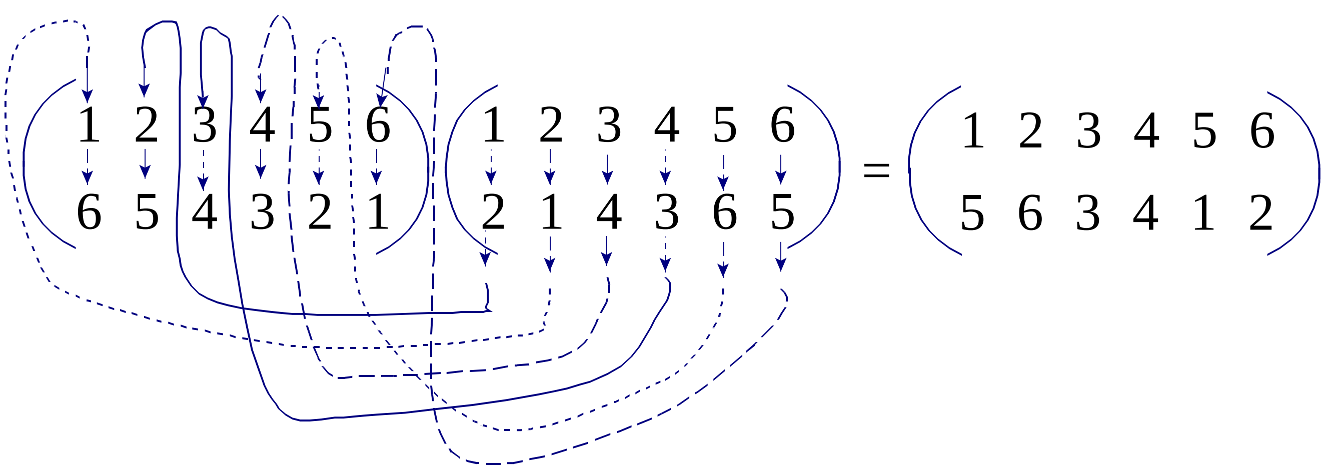
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$; $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$; $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix}$

1. $\tau\mu =$



i.e., $\tau\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix}$

2. $\sigma\tau =$



i.e., $\tau\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}$

Recall:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}; \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}; \qquad \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix}$$

3. $\mu^{-1} =$

The best way to do this is to reason that the **inverse** of a permutation sends the permuted elements "back to their original place." (i.e., "back to where they came from.")

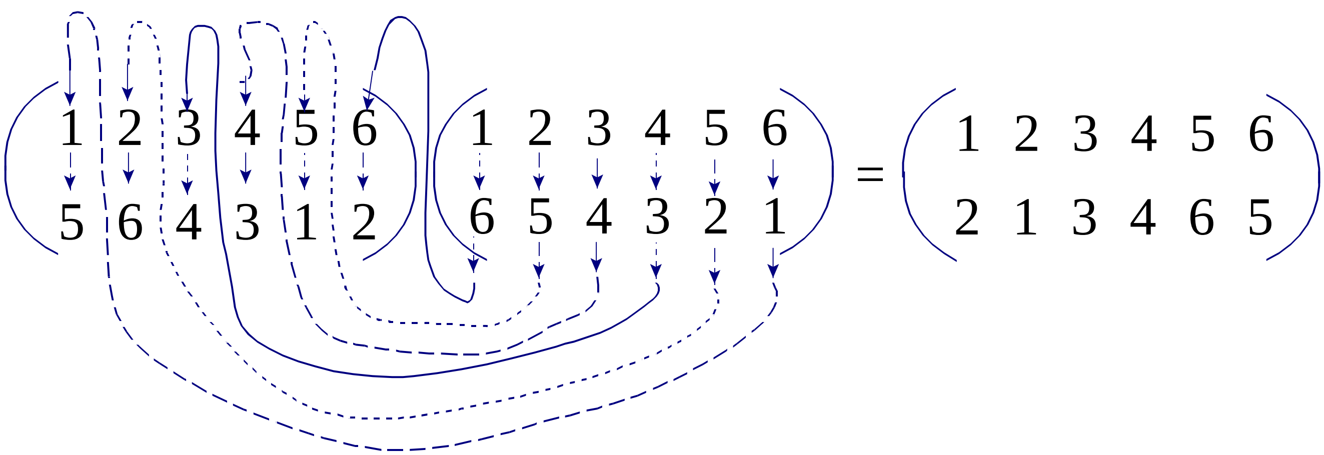
$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix}$$

The permutation σ maps $1 \to 5$, $2 \to 6$, $3 \to 4$, $4 \to 3$, $5 \to 1$, and $6 \to 2$.

So the permutation σ^{-1} maps $5 \to 1$, $6 \to 2$, $4 \to 3$, $3 \to 4$, $1 \to 5$, and $2 \to 6$.

Hence, $\mu^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix}$

4. $\mu^{-1}\sigma =$



Hence, $\mu^{-1}\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 6 & 5 \end{pmatrix}$

In Exercises 5-6, decompose the given permutations into the "product" of disjoint cycles.

Remark: The procedure for this is rather straightforward:

Start with the element "1" and follow this element through repeated applications of the permutation until you return to the element "1." The elements that are produced, as well as the order in which these elements appear, define one cycle.

Select the first element of the set that does not appear in the previous cycle. Follow this element through repeated applications of the permutation until you return to the element. The elements that are produced, as well as the order in which these elements appear, define the next cycle.

Continue with this process, until you have exhausted all elements.

5. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 6 & 5 & 8 & 3 & 1 & 2 \end{pmatrix}$

   Observe: $\pi(1) = 7$

   $\pi^2(1) = \pi(\pi(1)) = \pi(7) = 1$

   Thus, $(1, 7)$ is one cycle.

   The element "2" is the first element that does not appear in the previous cycle. So apply π repititively to "2."

   $\pi(2) = 4$

   $\pi^2(2) = \pi(\pi(2)) = \pi(4) = 5$

   $\pi^3(2) = \pi\left(\pi^2(2)\right) = \pi(5) = 8$

   $\pi^4(2) = \pi\left(\pi^3(2)\right) = \pi(8) = 2$

   Thus, $(2, 4, 5, 8)$ is the next cycle.

   The element "3" is the first element that does not appear in the previous cycle. So apply π repititively to "3."

   $\pi(3) = 6$

   $\pi^2(3) = \pi(\pi(3)) = \pi(6) = 3$

   Thus, $(3, 6)$ is one cycle.

   This exhausts all elements.

   i.e., $\sigma = (3,6)(2,4,5,8)(1,7) = (3,6)(1,7)(2,4,5,8) = (2,4,5,8)(3,6)(1,7) = (2,4,5,8)(1,7)(3,6)$

   $= (1,7)(3,6)(2,4,5,8) = (1,7)(2,4,5,8)(3,6)$

   Note: Since the cycles are **disjoint,** the **order** of the cycles is **arbitrary,** because **disjoint cycles commute.**

6. $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 6 & 2 & 7 & 5 & 8 & 3 \end{pmatrix}$

Observe: $\theta(1) = 4$

$\theta^2(1) = \theta(\theta(1)) = \theta(4) = 2$

$\theta^3(1) = \theta\left(\theta^2(1)\right) = \theta(2) = 1$

Thus, $(1, 4, 2)$ is one cycle.

The element "3" is the first element that does not appear in the previous cycle. So apply π repititively to "3."

$\theta(3) = 6$

$\theta^2(3) = \theta(\theta(3)) = \theta(6) = 5$

$\theta^3(3) = \theta\left(\theta^2(3)\right) = \theta(5) = 7$

$\theta^4(2) = \theta\left(\theta^3(2)\right) = \theta(7) = 8$

$\theta^5(2) = \theta\left(\theta^4(2)\right) = \theta(8) = 3$

Thus, $(3, 6, 5, 7, 8)$ is the next cycle.

This exhausts all elements.

i.e., $\sigma = (3, 6, 5, 7, 8)(1, 4, 2) = (1, 4, 2)(3, 6, 5, 7, 8)$

In Exercises 7-8, decompose the given permutations into the "product" of transpositions.

Recall: The "3-cycle" (a, b, c) can be decomposed into the transpositions $(a, b)\,(b, c)$. (i.e., $(a, b, c) = (a, b)\,(b, c)$)

Similarly: The "n-cycle" $(x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n) = (x_1, x_2)\,(x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$ can be decomposed into $(x_1, x_2)\,(x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$.

(i.e., $(x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n) = (x_1, x_2)\,(x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$)

7. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 6 & 5 & 8 & 3 & 1 & 2 \end{pmatrix}$

   Recall: $\pi = (3, 6)\,(2, 4, 5, 8)\,(1, 7)$

   Thus, $\pi = (3, 6)\,(2, 4, 5, 8)\,(1, 7) = (3, 6)\,\underbrace{(2, 4)\,(4, 5, 8)}_{=(2,4,5,8)}\,(1, 7) = (3, 6)\,(2, 4)\,\underbrace{(4, 5)\,(5, 8)}_{=(4,5,8)}\,(1, 7)$

   $$\boxed{\pi = (3, 6)\,(2, 4)\,(4, 5)\,(5, 8)\,(1, 7)}$$

8. $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 6 & 2 & 7 & 5 & 8 & 3 \end{pmatrix}$

   Recall: $\theta = (3, 6, 5, 7, 8)\,(1, 4, 2) = \underbrace{(3, 6)\,(6, 5, 7, 8)}_{=(3,6,5,7,8)}\underbrace{(1, 4)\,(4, 2)}_{=(1,4,2)} = (3, 6)\,\underbrace{(6, 5)\,(5, 7, 8)}_{=(6,5,7,8)}\,(1, 4)\,(4, 2) =$
   $(3, 6)\,(6, 5)\,\underbrace{(5, 7)\,(7, 8)}_{=(5,7,8)}\,(1, 4)\,(4, 2)$

   $$\boxed{\theta = (3, 6)\,(6, 5)\,(5, 7)\,(7, 8)\,(1, 4)\,(4, 2)}$$