MTH 6610 History of Math - Midterm Exam

Spring 2016

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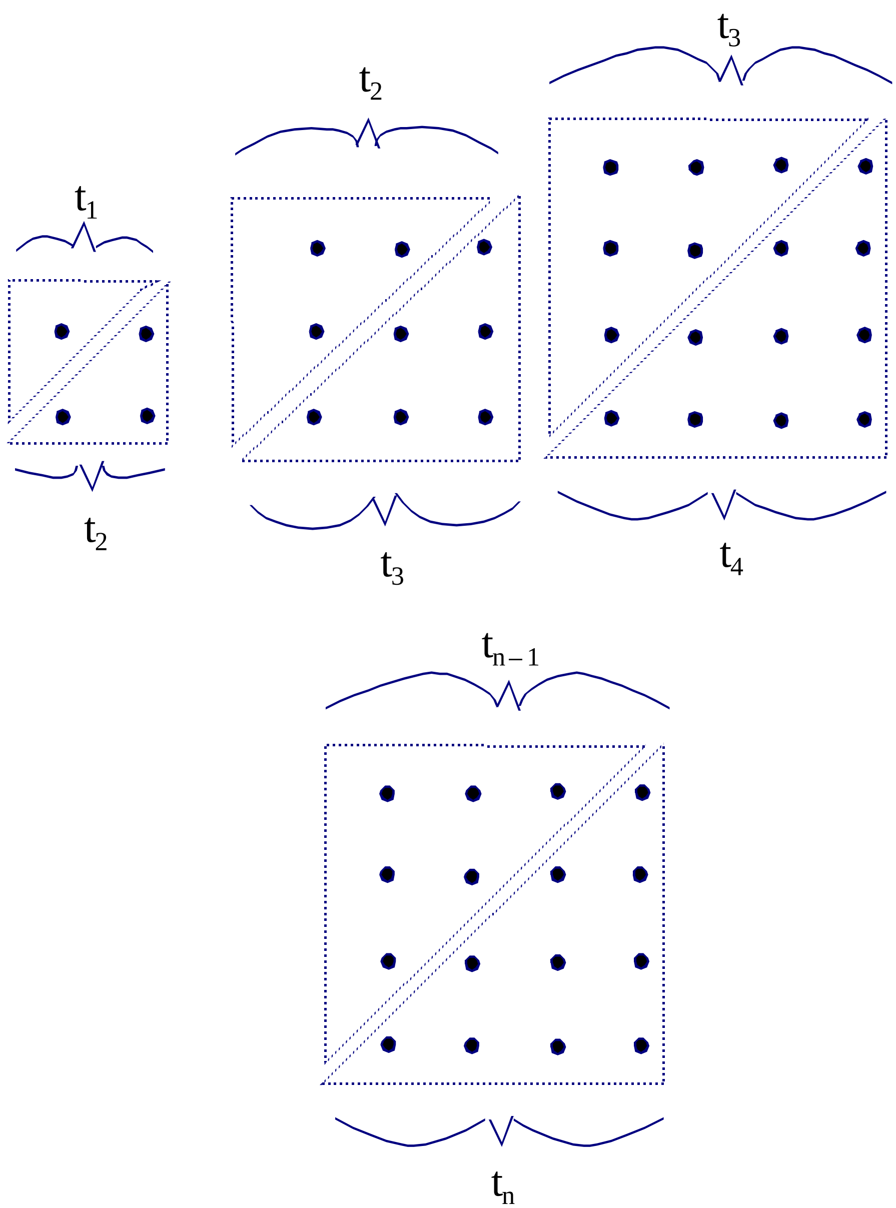
Instructions. Show CLEARLY how you arrive at your answers

1. Using dot diagrams, show that the sum of consecutive triangular numbers is a square number (in particular, show that $t_{n-1} + t_n = s_n$).

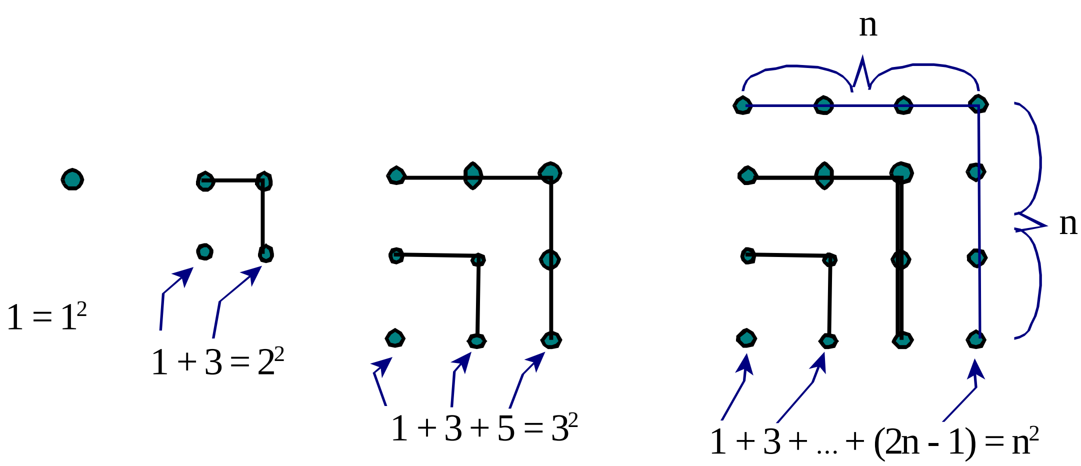


The first three diagrams illustrate that $t_1 + t_2 = s_2$; $t_2 + t_3 = s_3$; and $t_3 + t_4 = s_4$. The diagram on the bottom shows that s_n can be decomposed into $t_{n-1} + t_n$.

2. Using dot diagrams, show that the sum of the first n odd natural numbers is equal to n^2.

   i.e., show that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

   Consider: The "picture proof" below.



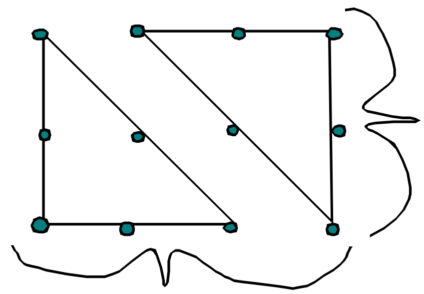
   Observe that, in the first three diagrams, we illustrate that:

   $\sum_{i=1}^{1}(2i-1) = 1^2$ $\quad$ $\sum_{i=1}^{2}(2i-1) = 2^2$ $\quad$ and $\sum_{i=1}^{3}(2i-1) = 3^2$

   In the last diagram, we illustrate the general case: $\sum_{i=1}^{n}(2i-1) = n^2$

3. Using dot diagrams, derive a formula for the value of the n^{th} triangular number, t_n.

   Observe: Regarding the triangular numbers, $t_n = 1+2+3+\ldots+n$ for $n = 1, 2, 3, \ldots$



   The diagram above shows two triangles, each representing t_n, combined to form a rectangle of dots – having n rows and $(n+1)$ columns of dots. Thus we have:

   $t_n + t_n = n\,(n+1)$

$\Rightarrow 2t_n = n\,(n+1)$

$\Rightarrow t_n = \frac{n(n+1)}{2}$

4. Illustrate the Babylonian method for generating Pythagorean Triples with three examples.

   We let $x = 2mn$; $z = m^2 + n^2$; and $y = m^2 - n^2$ (Note: $m > n$)

$m = 2$	$n = 1$

   This yields:

   $x = 2\,(2)\,(1) = 4$

   $z = (2)^2 + (1)^2 = 5$

   $y = (2)^2 - (1)^2 = 3$

$m = 3$	$n = 1$

   This yields:

   $x = 2\,(3)\,(1) = 6$

   $z = (3)^2 + (1)^2 = 10$

   $y = (3)^2 - (1)^2 = 8$

$\boxed{m = 3 \qquad n = 2}$ This yields:

$x = 2\,(3)\,(2) = 12$

$z = (3)^2 + (2)^2 = 13$

$y = (3)^2 - (2)^2 = 5$

5. Illustrate the Pythagoreans' method for generating Pythagorean Triples with three examples

   We let $x = 2n + 1; \quad y = 2n^2 + 2n; \quad z = 2n^2 + 2n + 1$

   $\boxed{n = 1}$ This yields:

   $x = 2\,(1) + 1 = 3$

   $y = 2\,(1)^2 + 2\,(1) = 4$

   $z = 2\,(1)^2 + 2\,(1) + 1 = 5$

   $\boxed{n = 2}$ This yields:

   $x = 2\,(2) + 1 = 5$

   $y = 2\,(2)^2 + 2\,(2) = 12$

   $z = 2\,(2)^2 + 2\,(2) + 1 = 13$

   $\boxed{n = 3}$ This yields:

   $x = 2\,(3) + 1 = 7$

   $y = 2\,(3)^2 + 2\,(3) = 24$

   $z = 2\,(3)^2 + 2\,(3) + 1 = 25$

6. Illustrate Plato's method for generating Pythagorean Triples with three examples

We let $x = 2n$; $y = n^2 - 1$; $z = n^2 + 1$; $n > 1$

$\boxed{n = 2}$ This yields:

$x = 2\,(2) = 4$

$y = (2)^2 - 1 = 3$

$z = (2)^2 + 1 = 5$

$\boxed{n = 3}$ This yields:

$x = 2\,(3) = 6$

$y = (3)^2 - 1 = 8$

$z = (3)^2 + 1 = 10$

$\boxed{n = 4}$ This yields:

$x = 2\,(4) = 8$

$y = (4)^2 - 1 = 15$

$z = (4)^2 + 1 = 17$

From problems 7-8, select one.

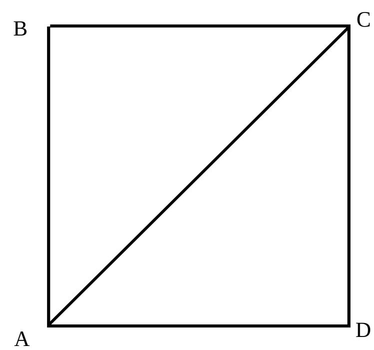
7. Prove that the side and diagonal of a square are "incommensurable" (not of common measure)

   Proof:

   Suppose, for the sake of deriving a contradiction, that the side and the diagonal of a square are "commensurable."

   (i.e., Suppose that there exists an increment δ and integers m and n such that the length of the diagonal $= m\delta$ and the length of a side $= n\delta$.)

Consider the square ABCD, and side AB and diagonal AC.



Then, by our hypothesis, there exist an increment δ and integers m and n such that $AC = m\delta$ and $AB = n\delta$.

Hence the ratio $\frac{AC}{AB} = \frac{m}{n}$, which is rational (because it's the quotient of integers).

Suppose also, without loss of generality, that m and n are relatively prime.

Then, we have $\frac{(AC)^2}{(AB)^2} = \frac{m^2}{n^2}$

By Pythagoreans Theorem, $(AB)^2 + (BC)^2 = (AC)^2$

$\Rightarrow (AB)^2 + (AB)^2 = (AC)^2$ (Because $ABCD$ is a *square.)*

$\Rightarrow 2\,(AB)^2 = (AC)^2$

$\Rightarrow 2 = \frac{(AC)^2}{(AB)^2} = \frac{m^2}{n^2}$

i.e., $2 = \frac{m^2}{n^2}$

$\Rightarrow 2n^2 = m^2$ (eq. 1)

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even

$\Rightarrow m = 2k$ for some $k \in \mathbf{N}$

$\Rightarrow$ (from eq.1), $2n^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$

$\Rightarrow n^2$ is even

$\Rightarrow n$ is even

$\Rightarrow n = 2j$ for some $j \in \mathbf{N}$

i.e., $m = 2k$ and $n = 2j$

$\Rightarrow m$ and n are NOT relatively prime.

This is a contradiction. Since the assumption that the side and the diagonal of a square are "commensurable" leads to a contradiction, the assumption must be false.

Hence, the side and the diagonal of a square are "incommensurable."

8. Prove that $\sqrt{2}$ is irrational.

Assume, for the sake of deriving a contradiction, that $\sqrt{2}$ is rational. Then there exist integers m and n such that $\sqrt{2} = \frac{m}{n}$ which is rational.

Suppose also, without loss of generality, that m and n are relatively prime.

$\Rightarrow 2 = \frac{m^2}{n^2}$

$\Rightarrow 2n^2 = m^2 \quad$ (eq. 1)

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even

$\Rightarrow m = 2k$ for some $k \in \mathbf{N}$

$\Rightarrow$ (from eq. 1), $\quad 2m^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$

$\Rightarrow n^2$ is even

$\Rightarrow n$ is even

$\Rightarrow n = 2j$ for some $j \in \mathbf{N}$

i.e., $m = 2k$ and $n = 2j$

$\Rightarrow m$ and n are NOT relatively prime.

This is a contradiction. Since the assumption that $\sqrt{2}$ is rational leads to a contradiction, this assumption must be false.

Hence, $\sqrt{2}$ is irrational.

9. Show (and explain) how Thales measured the height of the Great Pyramid.

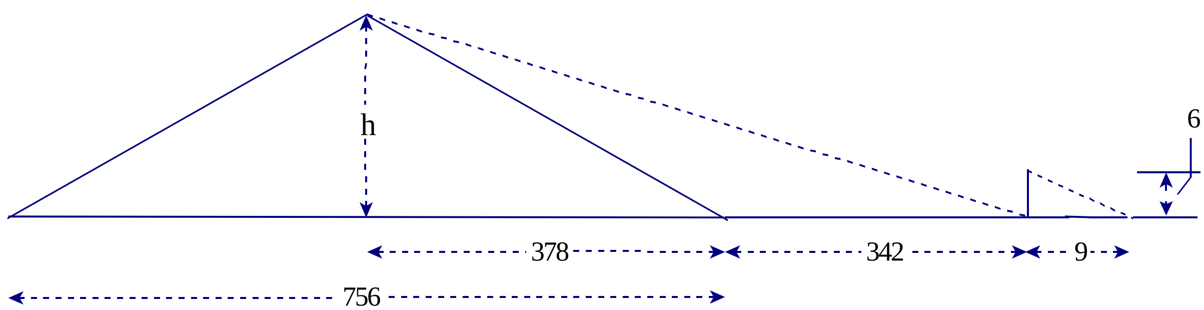
   Referring to the picture below:

   Thales measured the length of the base of one side of the pyramid and found the length to be (the equivalent of) 756 feet.

   Hence, the horizontal distance of the top of the pyramid to the side of the pyramid would be (the equivalent of) 378 feet.

   At a particular time of day, Thales measured the length of the shadow cast by the pyramid. From the base of the pyramid, the shadow continued for another 342 feet.

   At the end of the shadow, Thales planted a staff (6 feet in height) and measured the length of the shadow cast by the staff (9 feet).



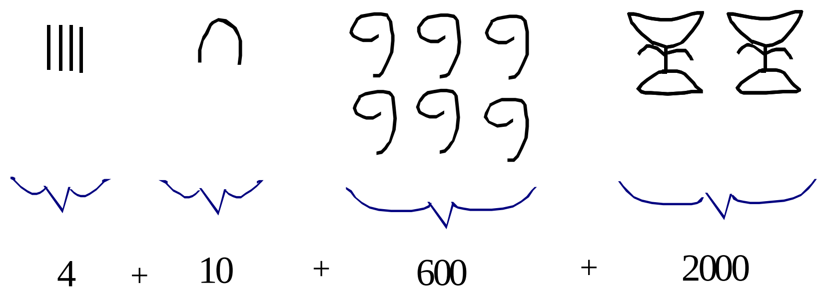
   By similar triangles, we have: $\frac{h}{378+342} = \frac{6}{9}$

   $\Rightarrow h = 480$ ft

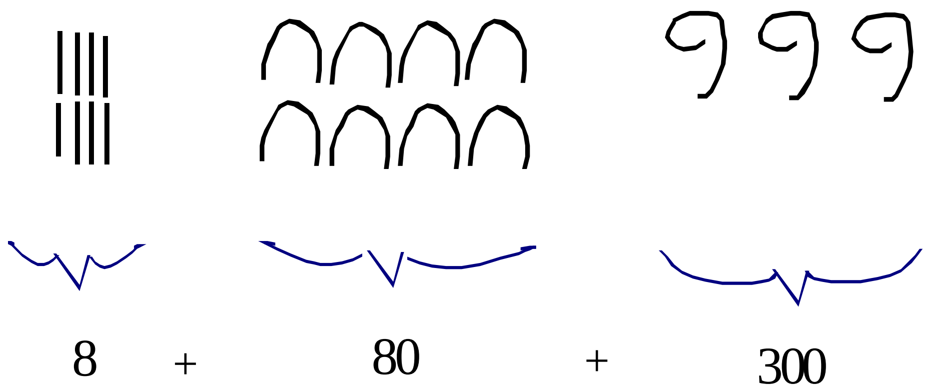
From the Homework

10. Convert from Hindu-Arabic (our number system) to Egyptian hieroglyphics:

    (a) $2614 = 2000 + 600 + 10 + 4 =$

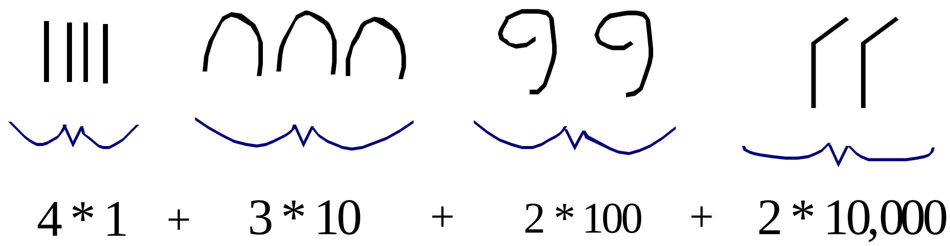


    (b) $388 = 300 + 80 + 8 =$



11. Convert the Egyptian numbers to Arabic (our number system).

(a) ‘



$= 20,000 + 200 + 30 + 4 = 20,234$

12. Write the Ionian Greek numerals corresponding to:

(a) $2643 = 2000 + 600 + 40 + 3 = \,,\beta\chi\mu\gamma$

(b) $876 = 800 + 70 + 6 = \omega o$ϛ

13. Solve this equation by the "Egyptian method." (i.e. False Position) $x + \frac{1}{5}x = 14$

First, make a "strategic guess" (e.g. $x = 5$), and plug this value into the equation.

$(5) + \frac{1}{5}(5) = 6$

Multiply both sides by whatever is necessary to make thr RH side equal to what it *should* be equal to. (Typically, this will be $\frac{c}{a+b}$)

$\frac{14}{6}\left[(5) + \frac{1}{5}(5)\right] = \frac{14}{6}[6]$

Distribute this factor on the LH side in such a way that the equation retains its original form ($x + \frac{1}{5}x = 14$)

$\Rightarrow \left[\left(\frac{70}{6}\right) + \frac{1}{5}\left(\frac{70}{6}\right)\right] = 14$

Since the equation is in its original form, the value in parentheses $\left(\frac{70}{6}\right)$ must be the value of x.

$x = \frac{70}{6} = \frac{35}{3}$

14. Solve this equation by the "Egyptian method." (i.e. Double False Position) $6x+8=0$

    Make two "random guesses," g_1 and g_2, for the solution.

    $g_1 = 1 \qquad g_2 = 10$

    Considering the LH side of the equation to be $f(x)$, compute the values of $f(x)$ at $x = g_1$ and $x = g_2$, respectivley.

    $f_1 = f(g_1) = f(1) = 6(1) + 8 = 14$

    $f_2 = f(g_2) = f(10) = 6(10) + 8 = 68$

    The solution is $x = \frac{f_1 g_2 - f_2 g_1}{f_1 - f_2} = \frac{(14)(10)-(68)(1)}{14-68} = -\frac{4}{3}$

    i.e., $x = -\frac{4}{3}$

15. Using the 2/n table, write $\frac{4}{11}$ as the sum of unit fractions (with no repetition)

    $\frac{4}{11} = \frac{2}{11} + \frac{2}{11} = \left(\frac{1}{6} + \frac{1}{66}\right) + \left(\frac{1}{6} + \frac{1}{66}\right) = \frac{2}{6} + \frac{2}{66} = \frac{1}{3} + \frac{1}{33}$

    i.e., $\frac{4}{11} = \frac{1}{3} + \frac{1}{33}$

16. Represent $\frac{3}{7}$ as the sum of distinct unit fractions, by using the splitting identity

    The "splitting identity" is: $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$

    $\frac{3}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{1}{7} + \left(\frac{1}{7+1} + \frac{1}{7(7+1)}\right) + \left(\frac{1}{7+1} + \frac{1}{7(7+1)}\right)$

    $= \frac{1}{7} + \left(\frac{1}{8} + \frac{1}{56}\right) + \left(\frac{1}{8} + \frac{1}{56}\right) = \frac{1}{7} + 2\left(\frac{1}{8} + \frac{1}{56}\right)$

    $= \frac{1}{7} + \left(\frac{2}{8} + \frac{2}{56}\right) = \frac{1}{7} + \left(\frac{1}{4} + \frac{1}{28}\right) = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$

    i.e., $\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$

17. Represent $\frac{3}{7}$ as the sum of distinct unit fractions, by using Fibonacci's method.

Find the integer n satisfying the inequality

$\frac{1}{n} \leq \frac{3}{7} < \frac{1}{n-1}$

$n = 3$ works

$\Rightarrow \frac{1}{3} \leq \frac{3}{7} < \frac{1}{2}$

Observe:

$0 \leq \frac{3}{7} - \frac{1}{3} = \frac{2}{21}$

$\Rightarrow \frac{3}{7} = \frac{1}{3} + \frac{2}{21} \qquad \text{(eq.1)}$

Repeat the same procedure with $\frac{2}{21}$

Find n such that $\frac{1}{n} \leq \frac{2}{21} < \frac{1}{n-1}$

$n = 11$ works

$\Rightarrow \frac{1}{11} \leq \frac{2}{21} < \frac{1}{10}$

$0 \leq \frac{2}{21} - \frac{1}{11} = \frac{1}{231}$

$\Rightarrow \frac{2}{21} = \frac{1}{11} + \frac{1}{231}$

Recall: From eq. 1, we have:

$\frac{3}{7} = \frac{1}{3} + \frac{2}{21}$

$\Rightarrow \frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$

i.e., $\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$

For problems 18 - 19, use "Egyptian methods" to compute:

18. $47 \div 8$ (Do this out "long division" the whole way)

$$\underbrace{47}_{\text{dividend}} \div \underbrace{8}_{\text{divisor}}$$

*	1	8
	2	16
*	4	32
*	$\frac{1}{2}$	4
*	$\frac{1}{4}$	2
*	$\frac{1}{8}$	1
Totals	$5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	

i.e., $47 \div 8 = 5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

19. $26 \cdot 33$

$$\underbrace{26}_{\text{multiplier}} \cdot \underbrace{33}_{\text{multiplicand}}$$

	1	33
*	2	**66**
	4	132
*	8	**264**
*	16	**528**
Totals	**26**	**858**

i.e., $26 \cdot 33 = 858$

20. Verify that $(3, 4, 5)$ is the only Pythagorean Triple involving consecutive positive numbers.

Suppose that $(n, n+1, n+2)$ is a pythagorean triple.

Then $n^2 + (n+1)^2 = (n+2)^2$

$\Rightarrow n^2 + (n^2 + 2n + 1) = n^2 + 4n + 4$

$\Rightarrow n^2 - 2n - 3 = 0$

$\Rightarrow (n+1)(n-3) = 0$

$\Rightarrow n = -1$; or $n = 3$

Since n must be a positive number, our sole solution is:

$n = 3$

$(n, n+1, n+2) = (3, 4, 5)$ is the only pythagorean triple involving consecutive positive numbers.

21. With the natural numbers from 1 to n as arranged below, derive a well known formula by adding the columns and the rows.

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \end{array}$$

Observe:

$$\begin{array}{ccccccc|c} 1 & 2 & 3 & \ldots & n-1 & n & = & (1+2+3+\ldots+n) \\ n & n-1 & n-2 & \ldots & 2 & 1 & = & (1+2+3+\ldots+n) \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & & \shortparallel \\ \hline (n+1) & (n+1) & (n+1) & (n+1) & (n+1) & (n+1) & = & 2\,(1+2+3+\ldots+n) \end{array}$$

From the table, we have: $n\,(n+1) = 2\,(1+2+3+\ldots+n)$

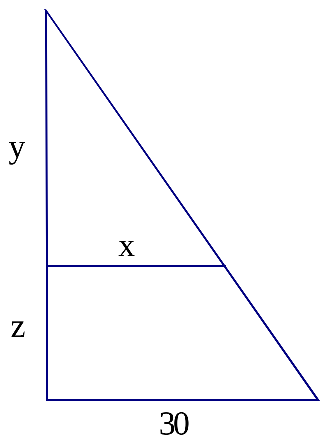
$\Rightarrow \frac{n(n+1)}{2} = 1+2+3+\ldots+n$

i.e., $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

22. A triangle whose base has length 30 is divided into two parts by a line segment drawn parallel to its base. It is given that the resulting right trapezoid has an area larger by 7,0 = 420 than the upper triangle, and that the difference between the height y of the upper trianlge and the height z of the trapezoid is 20. If x is the length of the upper base of the trapezoid these statements lead to the equations:

$$\frac{1}{2}z(x+30) = \frac{1}{2}xy + 420; \qquad y - z = 20$$

Find the quantities x, y, z using equivalent triangles.



From similar triangles, we have:

$\frac{y+z}{30} = \frac{y}{x}$

$\Rightarrow x = \frac{30y}{y+z}$ (eq. 1)

From the equation $y - z = 20$, we have:

$z = y - 20$ (eq. 2)

Substitute this for z in the eq. 1, and this yields:

$x = \frac{30y}{y+(y-20)}$

$\Rightarrow x = \frac{30y}{2y-20}$

$\Rightarrow x = \frac{15y}{y-10}$ (eq. 3)

Recall: $\frac{1}{2}z(x+30) = \frac{1}{2}xy + 420$

From eq. 2, $z = y - 20$. So we can substitute $y - 20$ for z in the previous equation, yielding:

$\Rightarrow \frac{1}{2}(y-20)(x+30) = \frac{1}{2}xy + 420$

$\Rightarrow (y-20)(x+30) = xy + 840$

$\Rightarrow 30y - 20x + xy - 600 = xy + 840$

$\Rightarrow 30y - 20x = 1440$

From eq. 3, we can substitute $\frac{15y}{y-10}$ for x in the previous equation, yielding:

$30y - 20\frac{15y}{y-10} = 1440$

$\Rightarrow 30y(y-10) - 20(15y) = 1440(y-10)$

$\Rightarrow 30y^2 - 300y - 300y = 1440y - 14400$

$\Rightarrow 30y^2 - 2040y + 14400 = 0$

$\Rightarrow y^2 - 68y + 480 = 0$

$\Rightarrow (y-8)(y-60) = 0$

$\Rightarrow y = 8; \qquad y = 60$

Note: since $z = y - 20$ must be a positive number, it follows that $y > 20$.

Hence, we discard the solution $y = 8$.

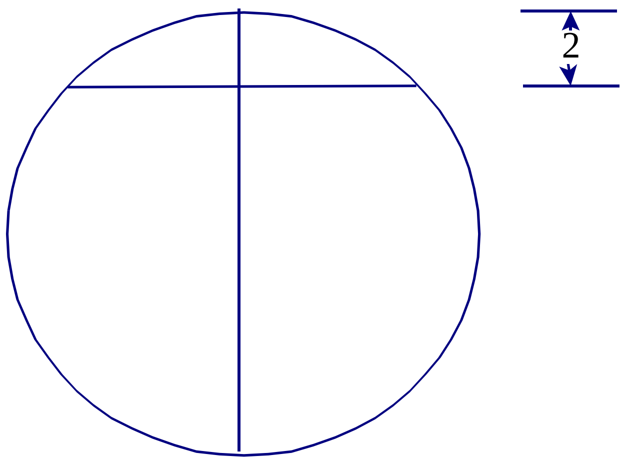
$\Rightarrow y = 60$

$\Rightarrow z = y - 20 = 40$

and $x = \frac{15y}{y-10} = \frac{15(60)}{(60)-10} = 18$

i.e., $y = 60; \qquad z = 40; \qquad x = 18$

23. Given that the circumference of a circle is 60 units and the length of a perpendicular from the center of a chord of the circle to the circumference is 2 units, find the length of the chord. In solving the problem, use $\pi = 3$, as did the Babylonians.



We are given that the Circumference is 60

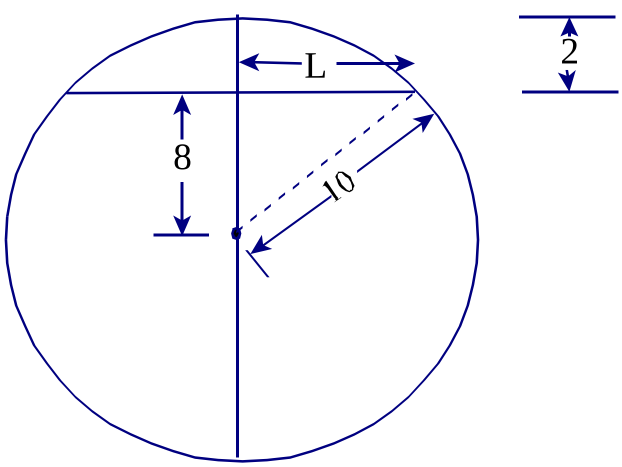
i.e., $C = 2\pi r = 60$

Using $\pi = 3$, this yields:

$C = 6r = 60$

$\Rightarrow r = 10$

If we let the length of the chord be $2L$, then we have the following:



By Pythagorean's Theorem, we have:

$L^2 + 8^2 = 10^2$

$\Rightarrow L^2 = 36$

$\Rightarrow L = 6$

Thus, the length of the chord is $2L = 12$