Mid-Term Exam Study Guide - Part #2

Spring 2017

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Thm - given sets A and B, with $|A|$ elements and $|B|$ elements, respectively, there are exactly $|B|^{|A|}$ functions from A to B.

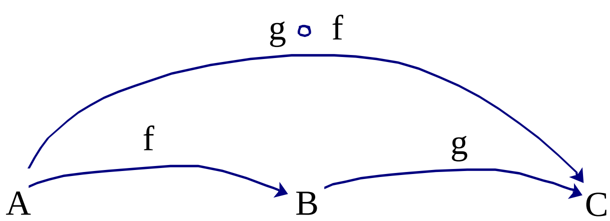
Proof. $\forall a \in A$, there are $|B|$ elements in B that a can be mapped to by f. So there are $(|B|$ choices for $f(a_1)) \cdot (|B|$ choices for $f(a_2)) \cdot \ldots \cdot (|B|$ choices for $f(a_n)) = |B|^n$ ■

Ex given that $A = \{a, b, c, d\}$ and $B = \{1, 2\}$, There exist $|B|^{|A|} = 2^4$ functions from $A \to B$.

Thm The composition of one to one functions is one to one.

Proof. Suppose that $f : A \to B$ and $g : B \to C$ are both one to one.

Then $g \circ f : A \to C$ as shown below.



To show that $g \circ f$ is one to one, we suppose that:

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ (Because g is one to one)

$\Rightarrow x_1 = x_2$ (Because f is one to one)

i.e., $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$

Hence, $(g \circ f)$ is one to one. ■

Thm The composition of one to one (onto) functions is one to one (onto)

(i.e., If $f : \mathbf{X} \to \mathbf{Y}$ is onto, and $g : \mathbf{Y} \to \mathbf{Z}$ is onto, then $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto.)

Proof. Let the hypothesis be given. (i.e., Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is onto, and $g : \mathbf{Y} \to \mathbf{Z}$ is onto.)

Let $z \in \mathbf{Z}$.

Since g is onto, $\exists y \in \mathbf{Y}$ (we'll call it y_z) such that $g(y_z) = z$.

Since f is onto, $\exists x \in \mathbf{X}$ (we'll call it x_z) such that $f(x_z) = y_z$.

Observe: $g(f(x_z)) = g(y_z) = z$.

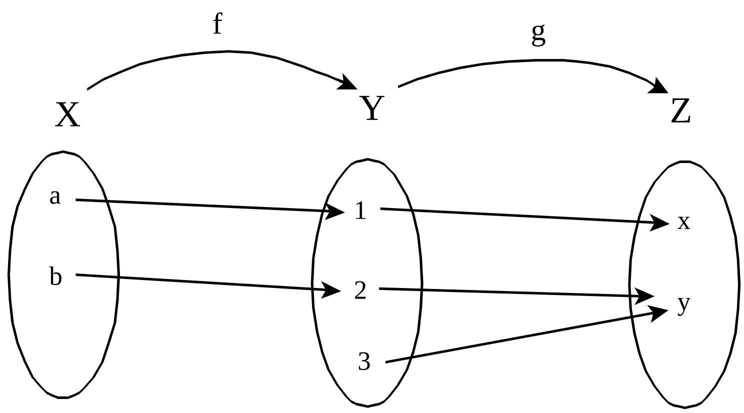
Thus, given $z \in \mathbf{Z}$, $\exists x \in \mathbf{X}$ such that $(g \circ f)(x) = z$.

Hence, $g \circ f$ is onto. ■

Proposition: Given $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$, Suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one. Is either f or g necessarily one to one?

Claim: g is not necessarily one to one.

Proof. Consider $f : \mathbf{X} \to \mathbf{Y}$ and g: $\mathbf{Y} \to \mathbf{Z}$ as shown below. Note that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one, as $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, but $g : \mathbf{Y} \to \mathbf{Z}$ is not one to one, as $f(2) = f(3)$.



Claim: f must be one to one.

Proof. Let the hypothesis be given. (i.e., suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one.)

Suppose also, for the sake of deriving a contradiction, that $f : \mathbf{X} \to \mathbf{Y}$ is not one to one. Then $\exists x_{1,}\ x_2 \in \mathbf{X}$, with $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$.

$\Rightarrow g(f(x_1)) = g(f(x_2))$.

Thus, $\exists x_{1,}\ x_2 \in \mathbf{X}$, with $x_1 \neq x_2$, such that $g(f(x_1)) = g(f(x_2))$.

$\Rightarrow g \circ f : \mathbf{X} \to \mathbf{Z}$ is not one to one, contrary to our hypothesis.

Since the assumption (that f is not one to one) yields a contradiction, it must be false.

Hence, $f : \mathbf{X} \to \mathbf{Y}$ is one to one.

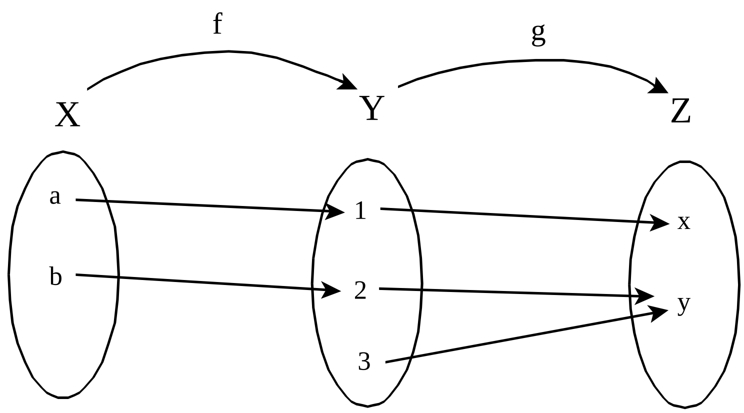
Thus, if $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one, then $f : \mathbf{X} \to \mathbf{Y}$ must be one to one , but $g : \mathbf{Y} \to \mathbf{Z}$ is not necessarily one to one. ∎

Thm (Relative to the previous proposition) - If $g \circ f$ is one to one, then f is one to one also.

Proposition: Given $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$, Suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto. Is either f or g necessarily onto?

Claim: f is not necessarily onto.

Proof. Consider $f : \mathbf{X} \to \mathbf{Y}$ and g: $\mathbf{Y} \to \mathbf{Z}$ as shown below. Note that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto, as $\forall z \in \mathbf{Z}$,$\exists x \in \mathbf{X}$ such that $g(f(x)) = z$, and yet $f : \mathbf{X} \to \mathbf{Y}$ is not onto.



Claim: $g : \mathbf{Y} \to \mathbf{Z}$ must be onto.

Proof. Let the hypothesis be given. (i.e., suppose that
$g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto.)Let $z \in \mathbf{Z}$ be given.

Then $\exists x \in \mathbf{X}$ such that $g(f(x)) = z$.

$\Rightarrow \exists y \in \mathbf{Y}$ (namely $y = f(x)$), such that $g(y) = z$.

i.e., Given $z \in \mathbf{Z}$, $\exists y \in \mathbf{Y}$ such that $g(y) = z$.

Hence, $g : \mathbf{Y} \to \mathbf{Z}$ must be onto.

Thus, if $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto, then $g : \mathbf{Y} \to \mathbf{Z}$ must be onto, but $f : \mathbf{X} \to \mathbf{Y}$is not necessarily onto. ■

Thm (Relative to the previous proposition) - If $g \circ f$ is onto, then g is onto.

Thm f has an inverse exactly when f is bijective

(i.e., A function $f : \mathbf{X} \to \mathbf{Y}$ has an inverse if and only if it is one to one and onto.)

Proof. If $f : \mathbf{X} \to \mathbf{Y}$ has an inverse then it is one to one and onto.

Let the hypothesis be given. (i.e., suppose that $f : \mathbf{X} \to \mathbf{Y}$ has an inverse, f^{-1})

Then: $f^{-1} \circ f = 1_X$, and hence, $f^{-1} \circ f$ is one to one and onto.

Also: $f \circ f^{-1} = 1_Y$, and hence, $f \circ f^{-1}$is one to one and onto.

Since $f^{-1} \circ f$ is one to one and onto, then by previous exercises (and), f must be one to one, and f^{-1} must be onto.

Since $f \circ f^{-1}$ is one to one and onto, then by previous exercises (and), f^{-1} must be one to one, and f must be onto.

Hence, both f and f^{-1} are one to one and onto.

In particular, f is one to one and onto.

If $f : \mathbf{X} \to \mathbf{Y}$ is one to one and onto, then it has an inverse.

Let the hypothesis be given. (i.e., Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is one to one and onto.)

Note that since f is onto, for any value of $y \in \mathbf{Y}$, there exists an $x \in \mathbf{X}$ such that $f(x) = y$.

Since f is one to one, there is *only* one $x \in \mathbf{X}$ such that $f(x) = y$. We'll call it x_y

Thus, we can define $f^{-1} : \mathbf{Y} \to \mathbf{X}$ by $f^{-1}(y) = x_y$

We must now check and make sure that $f^{-1} \circ f = 1_X$ and that $f \circ f^{-1} = 1_Y$.

Observe: $f^{-1} \circ f(x_y) = f^{-1}(f(x_y)) = f^{-1}(y) = x_y$.

Thus, $f^{-1} \circ f = 1_X$

Also: $f \circ f^{-1}(y) = f\left(f^{-1}(y)\right) = f(x_y) = y$

Hence, f has an inverse. ■

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Remark: Note that if $\exists f : \mathbf{X} \to \mathbf{Y}$ that is one to one and onto, then $\exists g : \mathbf{Y} \to \mathbf{X}$ that is one to one and onto (e.g., f^{-1})

Remark: If $\exists f : S \to \mathbf{N}$ that is one to one and onto, then $\exists g : \mathbf{N} \to \mathbf{S}$ that is one to one and onto also.

Remark: Thus, to show that a set is denumerable, we can show that $\exists g : \mathbf{N} \to \mathbf{S}$ that is one to one and onto, or we can show that $\exists f : S \to \mathbf{N}$ that is one to one and onto. Either is sufficient.

Remark: Since the composition of one to one and onto functions is also one to one and onto, if a set A is known to be denumerable, then any set B that can be put into a one to one correspondence with A is also denumerable.

The point is this: An alternate way of showing that a set B is denumerable is to exhibit a one to one correspondence between B and a set A, where A is known to be denumerable.

More Generally:

Remark: If $\exists\ f : A \to B$ (one to one and onto), then $\exists\ f^{-1} : B \to A$ (one to one and onto).

Hence, if $A \approx B$, then $B \approx A$.

Remark: If $\exists\ f : A \to B$ (one to one and onto) and $\exists\ g : B \to C$ (one to one and onto), then $(g \circ f) : A \to C$ is one to one and onto)

Thus, if $A \approx B$, and $B \approx C$, then $A \approx C$. Also, in light of an earlier observation, $C \approx A$.

Remark: In general, every set that has the same cardinality as A has the same cardinality as every other set that has the same cardinality as A.
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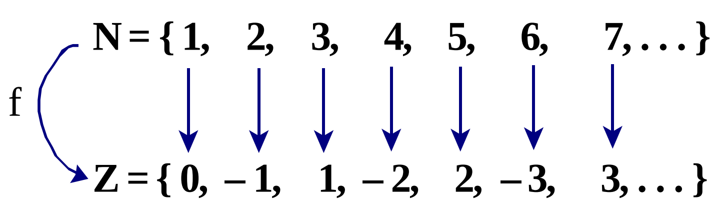
Thm $\mathbb{N} \approx \mathbb{Z}$ (i.e., $\mathbb{N}$ has the same cardinality as $\mathbb{Z}$.)

Proof. To show that $\mathbb{N}$ has the same cardinality as $\mathbb{Z}$, we must show that there exists a one to one correspondence between the two sets.

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \begin{cases} \frac{n-1}{2} & \text{for } n \text{ odd} \\ -\frac{n}{2} & \text{for } n \text{ even} \end{cases}$.

The function is pictured below:



Clearly, f is one to one and onto.

Hence, $\mathbb{N} \approx \mathbb{Z}$ ■

Exercise Prove that the function given in the previous proof is one to one and onto.

Proof. f maps the odd natural numbers one to one and onto the non-negative integers.

Suppose that $f(x_1) = f(x_2) \geq 0$

$\Rightarrow \frac{x_1-1}{2} = \frac{x_2-1}{2}$

$\Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2$

i.e., $f(x_1) = f(x_2) \geq 0 \Rightarrow x_1 = x_2$

Hence, f maps the odd natural numbers one to one into the non-negative integers.

Next, suppose that n is a non-negative integer and let $m = 2n + 1$.

(Note that m is an odd natural number.)

Observe that $f(m) = \frac{m-1}{2} = \frac{(2n+1)-1}{2} = \frac{2n}{2} = n$

i.e., for each non-negative integer n, $\exists$ an odd number m (namely, $m = 2n + 1$) such that $f(m) = n$.

f maps the even natural numbers one to one and onto the negative integers.

Suppose that $f(x_1) = f(x_2) < 0$

$\Rightarrow -\frac{x_1}{2} = -\frac{x_2}{2}$

$\Rightarrow -x_1 = -x_2$

$\Rightarrow x_1 = x_2$

i.e., $f(x_1) = f(x_2) < 0 \Rightarrow x_1 = x_2$

Hence, f maps the even natural numbers one to one into the negative integers.

Next, suppose that n is a negative integer and let $m = -2n$.

(Note that m is an even natural number.)

Observe that $f(m) = -\frac{m}{2} = -\frac{(-2n)}{2} = \frac{2n}{2} = n$

i.e., for each negative integer n, $\exists$ an even number m (namely, $m = -2n$) such that $f(m) = n$.

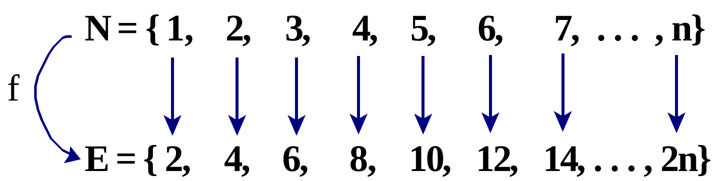
Thus, f maps $\mathbb{N}$ one to one and onto $\mathbb{Z}$. ■

Exercise Prove that $\mathbb{Z} \approx \mathbf{E}$

Proof. We will prove something a little easier to prove, namely $\mathbb{N} \approx \mathbf{E}$. Then, because of the preceding observations and the fact that $\mathbb{N} \approx \mathbb{Z}$, it will follow that $\mathbb{Z} \approx \mathbf{E}$.

Define: $f : \mathbb{N} \to \mathbf{E}$ given by $f(n) = 2n$.

The function is shown below:



Clearly, from the diagram, f maps $\mathbb{N}$ one to one and onto $\mathbb{Z}$. ∎

Thm: $\mathbf{Q}^+$ (The set of positive rational numbers) is denumerable.

Consider the table of ordered pairs below:

$$\begin{array}{ccccccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) & & (1,5) & \rightarrow & \dots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
(2,1) & & (2,2) & & (2,3) & & (2,4) & & (2,5) & \dots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & (3,5) & \dots & \\
 & \swarrow & & \nearrow & & & & & & & \\
(4,1) & & (4,2) & & (4,3) & & (4,4) & & (4,5) & \dots & \\
\downarrow & \nearrow & & & & & & & & & \\
(5,1) & & (5,2) & & (5,3) & & (5,4) & & (5,5) & \dots & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

If we consider the ordered pair (i, j) in the ith row and jth column to represent the quotient of integers $\frac{i}{j}$, then *every positive rational number appears in the table at least once.* (e.g., the rational number $\frac{m}{n}$ appears in the mth row and n^{th} column.)

Furthermore, the arrows in the table induce an **exhaustive ordering** of the positive rational numbers as follows:

$$1, \tfrac{1}{2}, 2, 3, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, 4, 5, \tfrac{1}{5}, \dots$$

(Note that we have discarded repetitions of rationals if they occur. e.g., we have discarded $(2, 2)$ because it is equivalent to $(1, 1)$ which is already on our list.)

Note also that since the positive rationals are **ordered,** they are in a one to one correspondence with the natural numbers.

Hence, the positive rational numbers are denumerable. ∎

Lemma The set of negative rational numbers $\mathbf{Q}^-$ is denumerable.

Proof. The function $f : \mathbf{Q}^+ \to \mathbf{Q}^-$ given by $f\left(\frac{m}{n}\right) = -\frac{m}{n}$ is clearly one to one and onto.

For if $f(x_1) = f(x_2)$,

Then $-x_1 = -x_2$

$\Rightarrow x_1 = x_2,$ thus f is one to one.

Also, given $y \in \mathbf{Q}^-$, we can choose $x \in \mathbf{Q}^+$, given by $x = -y$.

This yields $f(x) = -x = -(-y) = y$.

Thus, f is onto. ■

Lemma The union of a denumerable set and a finite set is denumerable (you can assume that the two sets are disjoint).

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, b_3, \ldots\}$.

Then A is finite and B is denumerable.

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_n & \text{if } n \leq k \\ b_{n-k} & \text{if } n > k \end{cases}$

The function f is shown below:

$$\begin{array}{ccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & \ldots, & k, & k+1, & k+2, & k+3, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \\ (A \cup B) & = & \{ & a_1, & a_2, & a_3, & \ldots, & a_k & b_1, & b_2, & b_3, & \ldots & \} \end{array}$$

Clearly from the diagram above, f is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

Thm The union of two (disjoint) denumerable sets is denumerable.

Proof. Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

The function f is shown below:

$$\begin{array}{cccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ (A \cup B) & = & \{ & a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & \ldots & \} \end{array}$$

Clearly from the diagram above, f is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

Thm The union of finitely many (disjoint) denumerable sets is denumerable (i.e., if $A_1, A_2, \ldots, A_n$ are denumerable, then $\cup_{i=1}^{n} A_i$ is denumerable.)

Proof. Suppose that $A_1, A_2, \ldots, A_n$ are denumerable. Then we can name their elements as follows:

$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$

$\vdots$

$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$

Consider:

$$\begin{array}{ccccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & & n, & n+1, & n+2, & & 2n, & 2n+1, & 2n+2, & & 3n, & \ldots \\ \uparrow & & & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow & \\ \cup_{i=1}^{n} A_i & = & \{ & a_{11}, & a_{21}, & \ldots, & a_{n1}, & a_{12}, & a_{22} & \ldots, & a_{n2}, & a_{13}, & a_{23}, & \ldots, & a_{n3}, & \ldots \end{array}$$

The function $f : \cup_{i=1}^{n} A_i \rightarrow \mathbf{N}$ given by $f(a_{ij}) = (j-1)\,n + i$ as shown above, is clearly one to one and onto. Hence, $\cup_{i=1}^{n} A_i$ is denumerable. ■

Alternate Proof

(By induction on n.)

(Step 1) Show that our proposition is true for $n = 1$

$\cup_{i=1}^{1} A_i = A_1$, which is denumerable, by hypothesis.

(Step 2) Assume that $\cup_{i=1}^{k} A_i$ is denumerable, and show that $\cup_{i=1}^{k+1} A_i$ is denumerable.

Observe: $\cup_{i=1}^{k+1} A_i = \left(\cup_{i=1}^{k} A_i\right) \cup A_{k+1}$, which is denumerable, since it is the union of two denumerable sets.

Hence, $\cup_{i=1}^{n} A_i$ is denumerable for all $n \in \mathbf{N}$. ■

Thm The union of denumerably many denumerable sets is denumerable (i.e., if A_1, $A_2, \ldots, A_n, \ldots$ are denumerable, then $\cup_{i=1}^{\infty} A_i$ is countable.) (Again, you can assume that the sets are disjoint.)

Proof. Let sets A_1, $A_2, \ldots, A_n, \ldots$ be denumerable and given by:

$$
\begin{array}{l}
A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\} \\
A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\} \\
A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\} \\
\vdots \\
A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\} \\
\vdots
\end{array}
$$

(Note that a_{ij} is the j^{th} element of the i^{th} set.)

Consider the table of elements from $\cup_{i=1}^{\infty} A_i$ listed below:

$$
\begin{array}{ccccccccccc}
a_{11} & \to & a_{12} & & a_{13} & \to & a_{14} & & a_{15} & \to & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & a_{25} & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & a_{35} & \cdots & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & a_{45} & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{51} & & a_{52} & & a_{53} & & a_{54} & & a_{55} & \cdots & \\
\vdots & \swarrow & \vdots & \nearrow & \vdots & \swarrow & \vdots & \nearrow & \vdots & &
\end{array}
$$

The table contains every element of $\cup_{i=1}^{\infty} A_i$. For example, the j^{th} element of set A_i is given by a_{ij}. This element is found in the i^{th} row and j^{th} column of the table.

Furthermore, the arrows in the table induce an exhaustive **ordering** of the elements of $\cup_{i=1}^{\infty} A_i$ as follows:

$$a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, a_{51}, a_{42}, a_{33}, a_{24}, a_{15}, \ldots$$

Note also that since the entire set of elements of $\cup_{i=1}^{\infty} A_i$ is **ordered,** they are in a one to one correspondence with the natural numbers.

Hence, the union of denumerably many denumerable sets is denumerable. ∎

Thm The set of rational numbers is denumerable (countable).

Proof. $\mathbf{Q}^+ \cup \{0\}$ is the union of a denumerable set and a finite set, hence it is denumerable.

The entire set of rationals can be expressed as $\mathbf{Q} = (\mathbf{Q}^+ \cup \{0\}) \cup \mathbf{Q}^-$, which is the union of two denumerable sets, hence denumerable. ∎

Note: All of the sets $\mathbb{N}, \mathbb{Z}, \mathbf{E}, \mathbb{Q}$ have the same cardinality. ($\mathbb{N} \approx \mathbb{Z} \approx \mathbf{E} \approx \mathbb{Q}$)

i.e., $|\mathbb{N}| = |\mathbb{Z}| = |\mathbf{E}| = |\mathbb{Q}|$ (or $n(\mathbb{N}) = n(\mathbb{Z}) = n(\mathbf{E}) = n(\mathbb{Q})$)

Lemma: The real numbers 0.5 and $0.499999\ldots$ are equal. (i.e., $0.5 = 0.4999\ldots$)

Proof. Suppose that $x = 0.4999\ldots$

Observe: $10x = 4.999\ldots$

Hence: $9x = 10x - x = (4.999\ldots) - (0.4999\ldots) = 4.5$

i.e., $9x = 4.5$

Hence, $x = 0.5$

But $x = 0.4999\ldots$ also.

Hence $0.5 = 0.4999\ldots$ ■

Remark: The previous proof hinges upon the supposition that we know how to add and subtract non-terminating decimals and that when we do, "things work out" just as we think they should.

Alternate Proof

Suppose, for the sake of deriving a contradiction, that $0.5 \neq 0.4999\ldots$

Then $0.5 > 0.4999\ldots$ and consequently, $\exists \varepsilon > 0$ such that $0.5 - 0.4999\ldots = \varepsilon$

By the Axiom of Archimedes, $\exists n \in \mathbf{N}$ such that $n > -\log(\varepsilon)$

Observe: Since $0.4999\ldots > \underbrace{0.4999}_{n \text{ decimal places}}$,

It follows that $\varepsilon = 0.5 - 0.4999\ldots < 0.5 - \underbrace{0.4999}_{n \text{ decimal places}} = 10^{-n} < 10^{\log(\varepsilon)} = \varepsilon$

Thus, $\varepsilon < \varepsilon$, a contradiction.

Since the assumption that $0.5 \neq 0.4999\ldots$ leads to a contradiction, the assumption must be false. Hence, $0.5 = 0.4999\ldots$ ■

Thm The set of real numbers in the interval $[0,1]$ is uncountable (non-denumerable).

Proof. (By contradiction)

Suppose, for the sake of deriving a contradiction, that the set of real numbers in the interval $[0,1]$ is denumerable.

Then there exists an *exhaustive ordering* of the set of real numbers in the interval $[0,1]$.

$$\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$$

Note that this ordering contains ALL of the real numbers in the interval $[0,1]$.

Consider the decimal expansions of these numbers:

$$\begin{aligned}
&x_1 = 0.x_{11}x_{12}x_{13}\ldots \\
&x_2 = 0.x_{21}x_{22}x_{23}\ldots \\
&x_3 = 0.x_{31}x_{32}x_{33}\ldots \\
&\vdots \\
&x_n = 0.x_{n1}x_{n2}x_{n3}\ldots x_{nn}\ldots \\
&\vdots
\end{aligned}$$

Observe: Here, x_{ij} is the jth digit past the decimal point in the decimal expansion of the ith real number x_i.

Also: If x_i can be written in terminating and non-terminating form (e.g., 0.5 can be written as $0.499999\ldots$), then we choose the non-terminating form.

(The number 0 will be represented as $0.000\ldots$)

Define $y \in [0,1]$ as follows:

$y = 0.y_1y_2y_3\ldots y_{n\ldots}$ where y_i is the ith digit past the decimal point in the decimal expansion of y.

For $n = 1, 2, 3, \ldots$ define the digit y_n as follows:

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5 \\ 1 & \text{if } x_{nn} = 5 \end{cases}$$

Observe: $y \in [0,1]$ and yet $y \neq x_n$ for any $n \in \mathbf{N}$.

The reason for this is that, by construction of y, the n^{th} digit of y is different from the n^{th} digit of x_n (i.e., $y_n \neq x_{nn}$) for all $n \in \mathbf{N}$.

Hence, $y \neq x_n \; \forall n \in \mathbf{N}$.

This contradicts our assumption that our list contains ALL of the real numbers in the interval $[0,1]$.

Since the assumption that the set of real numbers in the interval $[0,1]$ is denumerable led to this contradiction, the assumption must be false. Hence, the numbers in the interval $[0,1]$ is non-denumerable (uncountable). ∎

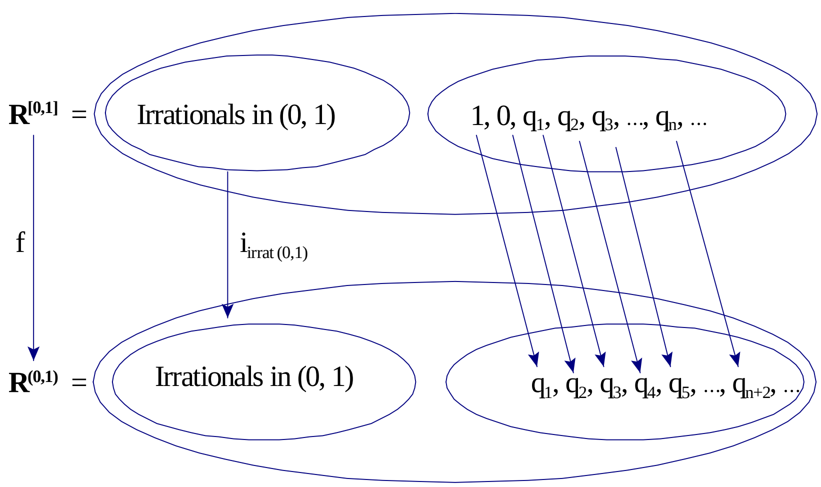
Proposition The set of real numbers in the interval $(0,1)$ has the same cardinality as the set of real numbers in the interval $[0,1]$. (i.e., $\exists f:[0,1]\to(0,1)$ that is one to one and onto.)

Proof. Let $\{q_1,q_2,q_3,\ldots,q_n,\ldots\}$ be an ordering of the rational numbers in the interval $(0,1)$. (Such an ordering exists, since the rationals in the interval $(0,1)$ are denumerable.)

Define $f:[0,1]\to(0,1)$ by

$$f(x)=\begin{cases} x & \text{if } x\in\mathbf{Q}^c \\ q_1 & \text{if } x=0 \\ q_2 & \text{if } x=1 \\ q_{n+2} & \text{if } x=q_n \text{ for } n\in\mathbf{N} \end{cases}$$

The function is shown graphically, below.



Clearly, $f:[0,1]\to(0,1)$ is one to one and onto. ∎

Corollary: The set of real numbers in the interval $(0,1)$ is non-denumerable (uncountable).

Remark: By assuming that $\mathbf{Q}^{(0,1)}$ (the set of rational numbers in the interval $(0,1)$) is denumerable, we have assumed the (intuitively) obvious fact that the subset of a denumerable set is denumerable (or finite). We prove this fact now.

Thm The subset of a denumerable set is denumerable (or finite).

(i.e., if A is a denumerable set and $B\subseteq A$, then either B is finite or B is denumerable)

Proof. Suppose that B is not finite.

Since A is denumerable, the elements of A can be ordered as follows:

$A=\{a_1,a_2,a_3,a_4,a_5,\ldots,a_n,\ldots\}$ (Eq. 1)

Let b_1 be the first element of A that appears as an element of B.

Similarly, let b_2 be the first element succeeding a_k in the sequence in Eq. 1 that is contained in B.

Proceeding inductively, we generate the entire set $B=\{b_1,b_2,b_3,b_4,b_5,\ldots,b_n,\ldots\}$. ∎

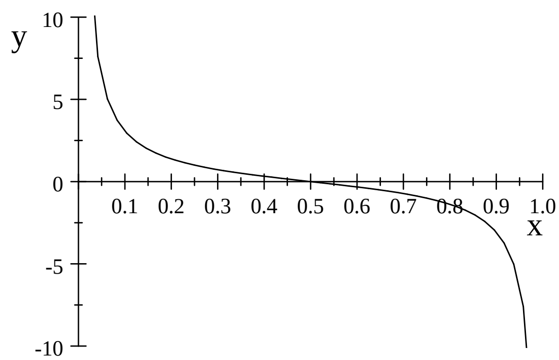
Remark: Regarding the previous proof, note that the Well-Ordering Principle guarantees the existence of a first element of A that appears as an element of B. If we let S be the set of subscripts of elements of A that are contained in B, then S is non-empty by our assumption that B is not finite. This means that S is a non-empty set of non-negative integers. Hence, S has a least element – call it k. Then a_k is the element that we call b_1.

The same reasoning can be used to guarantee the existence of $b_2, b_3, \ldots$

Thm The entire set of real numbers $\mathbf{R}$ is uncountable (non-denumerable).

Proof. Since the set of real numbers in the interval $(0,1)$ is non-denumerable, it suffices to exhibit a function $f:(0,1)\to\mathbf{R}$ that is one to one and onto.

Define $f:(0,1)\to\mathbf{R}$ by $f(x)=\cot(\pi x)$.



$f(x)=\cot(\pi x)$

Clearly, from the graph of $f(x)$, we see that $f(x)$ is one to one and onto. ■

Thm The set of irrational numbers $\mathbf{Q}^c$ is uncountable.

Proof. (By contradiction)

Suppose, for the sake of deriving a contradiction, that $\mathbf{Q}^c$ is countable (denumerable).

Then $\mathbf{R}=\mathbf{Q}\cup\mathbf{Q}^c$ is the union of two denumerable sets, hence, denumerable.

This contradicts the fact that $\mathbf{R}$ is uncountable (non-denumerable).

Since the assumption that $\mathbf{Q}^c$ is countable (denumerable) leads to a contradiction, $\mathbf{Q}^c$ must be uncountable (non-denumerable). ■

Proposition $\forall a, b \in \mathbf{R}$, with $a < b$, there is a one to one correspondence between the numbers in the interval $(0,1)$ and the numbers in the interval (a,b).

Proof. Let $a, b \in \mathbf{R}$, with $a < b$.

Consider $f : (a,b) \to (0,1)$ given by $f(x) = \frac{x-a}{b-a}$.

Observe: f is one to one, for if $f(x_1) = f(x_2)$,

then $\frac{x_1-a}{b-a} = \frac{x_2-a}{b-a}$

$\Rightarrow x_1 - a = x_2 - a$

$\Rightarrow x_1 = x_2$.

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, hence f is one to one.

Next, observe that f is onto, for given $y \in (0,1)$, there exists an $x \in (a,b)$, given by $x = \underline{y(b-a)+a}$ such that $f(x) = y$.

Observe: $f(x) = \frac{x-a}{b-a} = \underline{\frac{(y(b-a)+a)-a}{b-a}} = \underline{\frac{y(b-a)}{b-a}} = y$.

i.e., Given $y \in (0,1)$, there exists an $x \in (a,b)$, such that $f(x) = y$.

Therefore, f is onto. ■

Thm $|A| \prec |P(A)|$ (i.e., there does not exist a one to one and onto function from a set to its power set)

Proof. (By Contradiction) Let set A be given. Suppose, for the sake of contradiction, that there exists a function $f : A \to P(A)$ that is onto.

$\forall x \in A$, observe that not only is $f(x) \in P(A)$, but $f(x) \subseteq A$.

Define $x \in A$ to be a *good element* if $x \in f(x)$

Define $x \in A$ to be a *bad element* if $x \notin f(x)$.

Observe: Every element of A must either be a good element or a bad element (but not both).

Define $B \subseteq A$ to be the set of all bad elements of A.

i.e., $B = \{x \in A : x \notin f(x)\}$

Since $B \subseteq A$, it follows that $B \in f(A)$, by definition of power set.

Furthermore, since f is onto, by our contradiction assumption, $\exists b \in A$ such that $f(b) = B$.

Note that b cannot be a good element. For if b were a good element, then $b \in f(b)$

$\Rightarrow b \in B$ (because $f(b) = B$)

$\Rightarrow b \notin f(b)$, by definition of B.

i.e., $b \in f(b)$ and $b \notin f(b)$ - an impossibility.

Next, note that b cannot be a bad element. For if b were a bad element, then $b \notin f(b)$

$\Rightarrow b \notin B$ (because $f(b) = B$)

$\Rightarrow b \in f(b)$, by definition of B.

i.e., $b \notin f(b)$ and $b \in f(b)$ - an impossibility.

Therefore, b is neither a good element nor a bad element, contrary to our observation.

Hence, $f : A \to f(A)$ is not onto. ■

Thm The product of two denumerable sets is denumerable.

Proof. Let the sets be $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$

The product $A \times B$ is the set of all ordered pairs (a_i, b_j) with $a_i \in A$ and $b_j \in B$.

We will put all of these ordered pairs in a table such that all ordered pairs having a_i in the first coordinate are in row i and all ordered pairs having b_j in the second coordinate are in column j.

$$
\begin{array}{cccccccccc}
(a_1,b_1) & \longrightarrow & (a_1,b_2) & & (a_1,b_3) & \longrightarrow & (a_1,b_1) & & (a_1,b_5) & \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
(a_2,b_1) & & (a_2,b_2) & & (a_2,b_3) & & (a_2,b_4) & & (a_2,b_5) & \ldots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow \\
(a_3,b_1) & & (a_3,b_2) & & (a_3,b_3) & & (a_3,b_4) & & (a_3,b_5) & \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
(a_4,b_1) & & (a_4,b_2) & & (a_4,b_3) & & (a_4,b_4) & & (a_4,b_5) & \ldots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow \\
(a_5,b_1) & & (a_4,b_3) & & (a_4,b_3) & & (a_4,b_4) & & (a_4,b_5) & \ldots \\
\vdots & \swarrow & \vdots & \nearrow & \vdots & \swarrow & \vdots & \nearrow & \vdots & \swarrow
\end{array}
$$

We can draw arrows that induce an **exhaustive ordering** of the ordered pairs. Therefore, the set of ordered pairs, and hence the product $A \times B$, is denumerable. ■