MTH 1126 - Test #1A - Solutions

SPRING 2006

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\int_0^1 (3x^2-2)^4 \, x \, dx =$

   1. Is U-sub appropriate?

      a Is there a composite function?

      Yes! $\underbrace{\left(3x^2-2\right)}_{\text{inner}}\underbrace{{}^4}_{\text{outer}}$

      Let u be the "inner" function

      $\Rightarrow u = 3x^2 - 2$

      b Is there an approxiamte function/derivative pair?

      Yes! $\underbrace{\left(3x^2-2\right)}_{\text{function}} \rightarrow \underbrace{x}_{\text{deriv}}$

      Let u be the "function" of the function/derivative pair

      $\Rightarrow u = 3x^2 - 2$

   2. Compute du

$$\begin{array}{|rlcl|} \hline & u & = & 3x^2-2 \\ \Rightarrow & \frac{du}{dx} & = & 6x \\ \Rightarrow & du & = & 6xdx \\ \Rightarrow & \frac{1}{6}du & = & xdx \\ \hline \end{array}$$

$$\begin{array}{|lcl|} \hline \text{When } x=0, \quad u = 3x^2-2 & = & -2 \\ \\ \text{When } x=1, \quad u = 3x^2-2 & = & 1 \\ \hline \end{array}$$

   3. Analyze Integral in terms of u and du

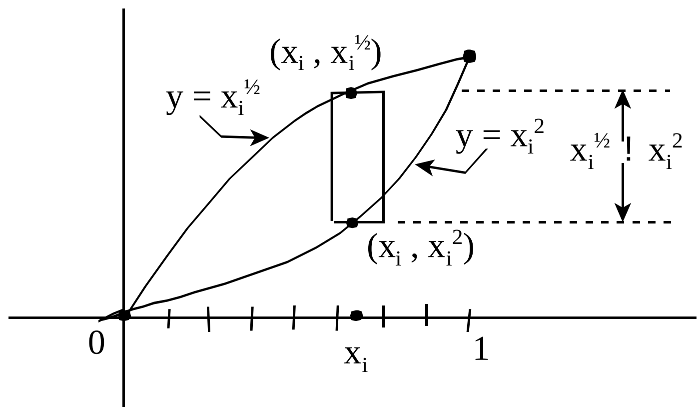
$$\int_{x=0}^{x=1} \underbrace{\left(3x^2-2\right)^4}_{u^4} \underbrace{x\,dx}_{\frac{1}{6}du} = \int_{u=-2}^{u=1} u^4 \tfrac{1}{6} du = \tfrac{1}{6}\int_{u=-2}^{u=1} u^4 du$$

   4. Integrate (in terms of u)

$$\tfrac{1}{6}\int_{u=-2}^{u=1} u^4 du = \tfrac{1}{6}\left[\tfrac{u^5}{5}\right]_{u=-2}^{u=1} = \tfrac{1}{6}\left[\left(\tfrac{1^5}{5}\right) - \left(\tfrac{(-2)^5}{5}\right)\right] = \tfrac{11}{10}$$

2. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 5$ and $g(x) = 2x - 2$

   1. First, graph the bounded region:



   Points of intersection: Set the y-coordinates equal to each other.

   $y = x^2 - 5 = 2x - 2$

   $\Rightarrow x^2 - 2x - 3 = 0$

   $\Rightarrow (x+1)(x-3) = 0$

   $\Rightarrow x = -1, 3$

   2. Observe: From $x = -1$ to $x = 3$, $2x - 2 \geq x^2 - 5$.

   Hence the bounded area is given by: $\int_{-1}^{3} [(2x-2) - (x^2-5)]\, dx$

   $= \int_{-1}^{3} (2x + 3 - x^2)\, dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^{3}$

   $= \left((3)^2 + 3(3) - \frac{1}{3}(3)^3\right) - \left((-1)^2 + 3(-1) - \frac{1}{3}(-1)^3\right) = \frac{32}{3}$

3. Suppose that $\int_2^8 (f(x) + g(x))\,dx = 9$; $\int_2^8 g(x)\,dx = 3$; and that $\int_2^4 f(x)\,dx = 5$. Compute $\int_4^8 f(x)\,dx$.

Since $\underbrace{\int_2^4 f(x)\,dx}_{\text{known}} + \underbrace{\int_4^8 f(x)\,dx}_{\text{what we want}} = \underbrace{\int_2^8 f(x)\,dx}_{\text{unknown}}$, our first order of business is to find $\int_2^8 f(x)\,dx$.

Observe: $\int_2^8 (f(x) + g(x))\,dx - \int_2^8 g(x)\,dx = \int_2^8 f(x)\,dx + \int_2^8 g(x)\,dx - \int_2^8 g(x)\,dx$

$= \int_2^8 f(x)\,dx$

i.e., $\int_2^8 f(x)\,dx = \underbrace{\int_2^8 (f(x) + g(x))\,dx}_{9} - \underbrace{\int_2^8 g(x)\,dx}_{3} = 9 - 3 = 6$

i.e., $\int_2^8 f(x)\,dx = 6$

Next observe: $\underbrace{\int_2^4 f(x)\,dx}_{5} + \int_4^8 f(x)\,dx = \underbrace{\int_2^8 f(x)\,dx}_{6}$

$\int_4^8 f(x)\,dx = \underbrace{\int_2^8 f(x)\,dx}_{6} - \underbrace{\int_2^4 f(x)\,dx}_{5} = 1$

i.e., $\int_4^8 f(x)\,dx = 6 - 5 = 1$

4. Compute: $\int \sin(x^3)\cos(x^3)\,x^2\,dx =$

   1. Is U-sub appropriate?

      a Is there a composite function?

      ??? $\underbrace{\sin}_{\text{outer}}\underbrace{\left(x^3\right)}_{\text{inner}}$; $\underbrace{\cos}_{\text{outer}}\underbrace{\left(x^3\right)}_{\text{inner}}$; these are both composite functions - How do we choose???

      b Is there an approxiamte function/derivative pair?

      Yes. $\underbrace{x^3}_{\text{function}} \rightarrow \underbrace{x^2}_{\text{deriv}}$ but ALSO $\underbrace{\sin\left(x^3\right)}_{\text{function}} \rightarrow \underbrace{\cos\left(x^3\right)x^2}_{\text{deriv}}$

      For now, we'll try the second pair.

      Let u be the "function" of the function/derivative pair

      $\Rightarrow u = \sin(x^3)$

   2. Compute du

      $$\begin{array}{|llcl|} \hline & u & = & \sin(x^3) \\ \Rightarrow & \frac{du}{dx} & = & \cos(x^3)\,3x^2 = 3x^2\cos(x^3) \\ \Rightarrow & du & = & 3x^2\cos(x^3)\ dx \\ \Rightarrow & \frac{1}{3}du & = & x^2\cos(x^3)\ dx \\ \hline \end{array}$$

   3. Analyze Integral in terms of u and du

      $\int \underbrace{\sin\left(x^3\right)}_{u}\underbrace{\cos\left(x^3\right)x^2\ dx}_{\frac{1}{3}du} = \int u \cdot \frac{1}{3}\ du = \frac{1}{3}\int u\ du$

   4. Integrate (in terms of u)

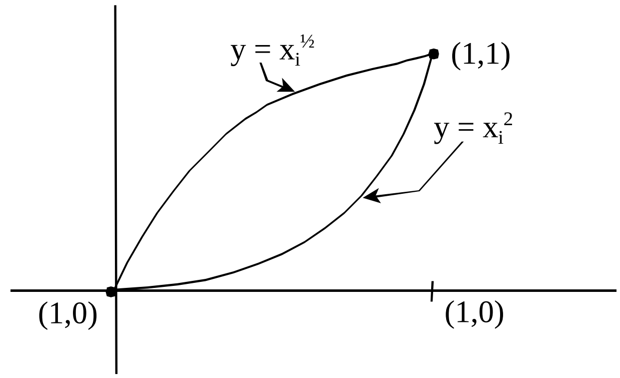
      $\frac{1}{3}\int u\ du = \frac{1}{3}\left[\frac{u^2}{2}\right] + C = \frac{1}{6}u^2 + C$

   5. Rewrite in terms of x

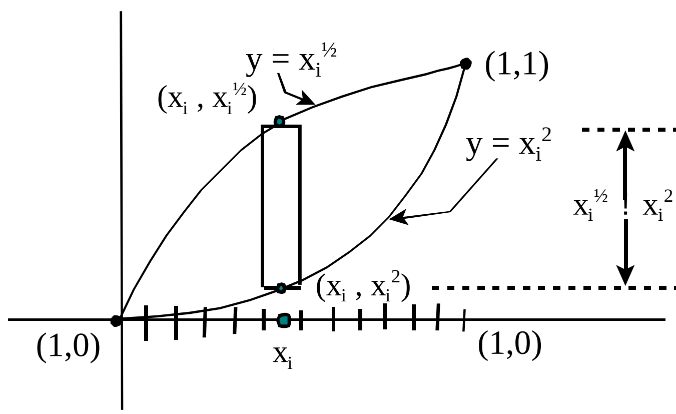
      $\int \sin(x^3)\cos(x^3)\,x^2\ dx = \frac{1}{6}\left(\sin(x^3)\right)^2 + C = \frac{1}{6}\sin^2(x^3) + C$

5. Find the area bounded by the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   1. Graph the bounded region and find the points of intersection:



   2. Partition the interval spanned by the region and inscribe a typical rectangle of width Δx over one of the rectangles.



   3. Compute the area of the i^{th} rectangle.

      Area of i^{th} rect $= \left(x_i^{\frac{1}{2}} - x_i^2\right) \Delta x$

   4. Approximate the area of the bounded region by adding up the areas of the rectangles

      Area $\approx \sum_{i=1}^{n} \left(x_i^{\frac{1}{2}} - x_i^2\right) \Delta x$

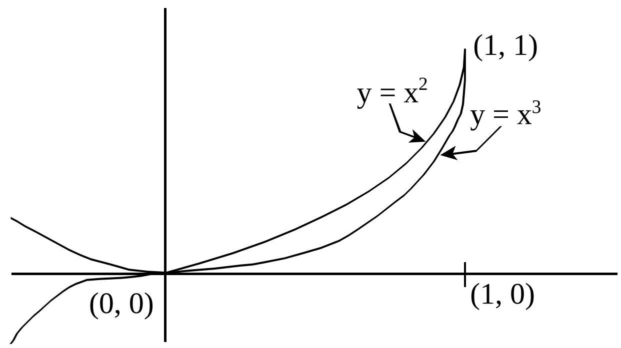
   5. Let $\Delta x \to 0$

      Area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(x_i^{\frac{1}{2}} - x_i^2\right) \Delta x = \int_{x=0}^{x=1} \left(x^{\frac{1}{2}} - x^2\right) dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$

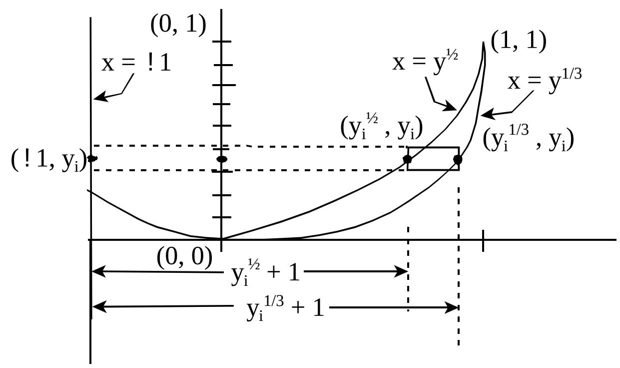
      $= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{3}(0)^3\right) = \frac{1}{3}$

6. A region in the x-y plane is bounded by the graphs $y = x^3$ and $y = x^2$. Use the Disk Method to compute the volume of the solid of revolution generated by revolving the region about the line $x = -1$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   1. Graph the bounded region and find the points of intersection:



   2. Draw a typical rectangle perpendicular to the axis of revolution and partition the interval spanned by the bounded region into sub-intervals of width Δx.



   3. Compute the volume of the i^{th} washer

   Vol. i^{th} washer = (Vol. i^{th} large disk) - (Vol. i^{th} hole)

   $$= \pi R_i^2 \Delta y - \pi r_i^2 \Delta y = \pi \left(y_i^{\frac{1}{3}} + 1 \right)^2 \Delta y - \pi \left(y_i^{\frac{1}{2}} + 1 \right)^2 \Delta y$$

   $$= \pi \left(y_i^{\frac{2}{3}} + 2y_i^{\frac{1}{3}} + 1 \right) \Delta y - \pi \left(y_i + 2y_i^{\frac{1}{2}} + 1 \right) \Delta y = \pi \left(y_i^{\frac{2}{3}} + 2y_i^{\frac{1}{3}} - y_i - 2y_i^{\frac{1}{2}} \right) \Delta y$$

   4. Approximate the volume of the solid of revolution by adding the volumes of the washers

   $$\text{Vol} \approx \sum_{i=1}^{n} \pi \left(y_i^{\frac{2}{3}} + 2y_i^{\frac{1}{3}} - y_i - 2y_i^{\frac{1}{2}} \right) \Delta y$$

5. Let $\Delta y \to 0$

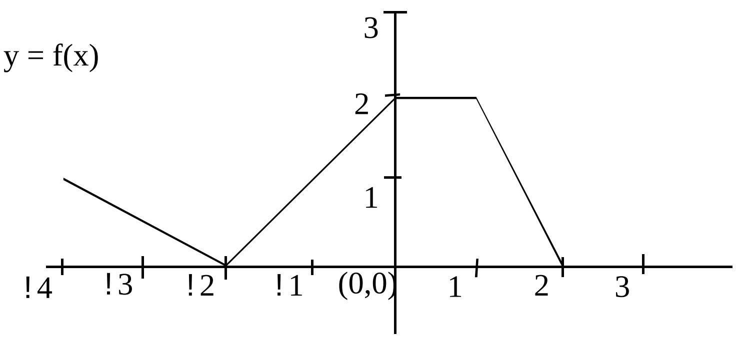
$\text{Vol} = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \pi \left(y_i^{\frac{2}{3}} + 2y_i^{\frac{1}{3}} - y_i - 2y_i^{\frac{1}{2}} \right) \Delta y = \int_{y=0}^{y=1} \pi \left(y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - y - 2y^{\frac{1}{2}} \right) dy$

$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} + \frac{3}{2} y^{\frac{4}{3}} - \frac{1}{2} y^2 - \frac{4}{3} y^{\frac{3}{2}} \right]_0^1$

$= \pi \left(\frac{3}{5} (1)^{\frac{5}{3}} + \frac{3}{2} (1)^{\frac{4}{3}} - \frac{1}{2} (1)^2 - \frac{4}{3} (1)^{\frac{3}{2}} \right) - \pi \left(\frac{3}{5} (0)^{\frac{5}{3}} + \frac{3}{2} (0)^{\frac{4}{3}} - \frac{1}{2} (0)^2 - \frac{4}{3} (0)^{\frac{3}{2}} \right)$

$= \frac{4\pi}{15}$

7. The graph of $f(x)$ is shown below. Compute $\int_{-3}^{3} f(x)\, dx$.



Since $f(x) \geq 0$ for $-3 \leq x \leq 3$, $\int_{-3}^{3} f(x)\, dx$ equals the area between the graph of $f(x)$ and the x-axis.

8. From the picture below, we have: $\int_{-3}^{3} f(x)\, dx = \frac{1}{4} + 2 + 2 + 1 = \frac{21}{4}$