MTH 1125 Test #1 - Solutions

Fall 2013 10 am class

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2} \frac{x^3+2}{x^2-9} =$

Step #1 Try Plugging In:

$$\lim_{x\to 2} \frac{x^3+2}{x^2-9} = \frac{(2)^3+2}{(2)^2-9} = \frac{10}{-5} = -2$$

$$\boxed{\text{i.e., } \lim_{x\to 2} \frac{x^3+2}{x^2-9} = -2}$$

2. Compute: $\lim_{x\to 6} \frac{x^2-5x-6}{x^2-3x-18} =$

Step #1 Try Plugging in:

$$\lim_{x\to 6} \frac{x^2-5x-6}{x^2-3x-18} = \frac{(6)^2-5(6)-6}{(6)^2-3(6)-18} = \frac{0}{0}$$

No Good - Zero Divide!

Step #2 Try Factoring and Cancelling:

$$\lim_{x\to 6} \frac{x^2-5x-6}{x^2-3x-18} = \lim_{x\to 6} \frac{(x-6)(x+1)}{(x-6)(x+3)} = \lim_{x\to 6} \frac{(x+1)}{(x+3)} = \frac{(6)+1}{(6)+3} = \frac{7}{9}$$

$$\boxed{\text{i.e., } \lim_{x\to 6} \frac{x^2-5x-6}{x^2-3x-18} = \frac{7}{9}}$$

3. Compute: $\lim_{x\to 4} \frac{x-5}{x^2-5x+4} =$

Step #1 Try Plugging in:

$$\lim_{x\to 4} \frac{x-5}{x^2-5x+4} = \frac{(4)-5}{(4)^2-5(4)+4} = \frac{-1}{0}$$

No Good - Zero Divide!

Step #2 Try Factoring and Cancelling:

No Good!. “Factoring and Cancelling” only works when Step #1 yields $\frac{0}{0}$.

Step #3 Analyze the one-sided limits:

$$\lim_{x\to 4^-} \frac{x-5}{x^2-5x+4} = \lim_{x\to 4^-} \frac{x-5}{(x-1)(x-4)} = \frac{-1}{(3)(-\varepsilon)} = \frac{\left(-\frac{1}{3}\right)}{(-\varepsilon)} = +\infty$$

$$\begin{array}{ll} & x \to 4^- \\ \Rightarrow & x < 4 \\ \Rightarrow & x - 4 < 0 \end{array}$$

$$\lim_{x\to 4^+} \frac{x-5}{x^2-5x+4} = \lim_{x\to 4^+} \frac{x-5}{(x-1)(x-4)} = \frac{-1}{(3)(+\varepsilon)} = \frac{\left(-\frac{1}{3}\right)}{(+\varepsilon)} = -\infty$$

$$\begin{array}{ll} & x \to 4^+ \\ \Rightarrow & x > 4 \\ \Rightarrow & x - 4 > 0 \end{array}$$

Since the one-sided limits are not equal, $\lim_{x\to 4} \frac{x-5}{x^2-5x+4}$ **Does Not Exist!**

4. $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x < 3 \\ 3x - 3 & \text{for } x \geq 3 \end{cases}$ Determine whether or not $f(x)$ is continuous at the point $x = 3$. (Justify your answer.)

If $f(x)$ is continuous at the point $x = 3$, then $\lim_{x\to 3} f(x) = f(3)$.

To see if this is true, we'll compute $\lim_{x\to 3} f(x)$.

Since the definiton of $f(x)$ changes at $x = 3$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^-} (x+3) = (3) + 3 = 6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$

Since the one-sided limits are equal, $\lim_{x\to 3} f(x)$ exists and $\lim_{x\to 3} f(x) = 6$

Furthermore, note that: $\lim_{x\to 3} f(x) = 6 = f(3)$

i.e., $\lim_{x\to 3} f(x) = f(3)$

Hence, $f(x)$ IS continuous at $x = 3$

5. $f(x) = \frac{x^2-x-6}{x^2+x-6}$ Find the asymptotes and graph

Verticals

1. Find x-values that cause division by zero.

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x+3)(x-2) = 0$

$\Rightarrow x = -3$ and $x = 2$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$$\lim_{x\to -3^-} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to -3^-} \frac{x^2-x-6}{(x+3)(x-2)} = \frac{6}{(-\varepsilon)(-5)} = \frac{6}{(\varepsilon)(5)} = \frac{\left(\frac{6}{5}\right)}{\varepsilon} = +\infty$$

$$\begin{array}{|ll|} & x \to -3^- \\ \Rightarrow & x < -3 \\ \Rightarrow & x+3 < 0 \\ \end{array}$$

$$\lim_{x\to -3^+} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to -3^+} \frac{x^2-x-6}{(x+3)(x-2)} = \frac{6}{(\varepsilon)(-5)} = \frac{\left(-\frac{6}{5}\right)}{\varepsilon} = -\infty$$

$$\begin{array}{|ll|} & x \to -3^+ \\ \Rightarrow & x > -3 \\ \Rightarrow & x+3 > 0 \\ \end{array}$$

Since the one-sided limits are infinite, $x = -3$ *is a vertical asymptote.*

$$\lim_{x\to 2^-} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to 2^-} \frac{x^2-x-6}{(x+3)(x-2)} = \frac{-4}{(5)(-\varepsilon)} = \frac{4}{(5)(\varepsilon)} = \frac{\left(\frac{4}{5}\right)}{(\varepsilon)} = +\infty$$

$$\begin{array}{|ll|} & x \to 2^- \\ \Rightarrow & x < 2 \\ \Rightarrow & x-2 < 0 \\ \end{array}$$

$$\lim_{x\to 2^+} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to 2^+} \frac{x^2-x-6}{(x+3)(x-2)} = \frac{-4}{(5)(\varepsilon)} = \frac{\left(-\frac{4}{5}\right)}{(\varepsilon)} = -\infty$$

$$\begin{array}{|ll|} & x \to 2^+ \\ \Rightarrow & x > 2 \\ \Rightarrow & x-2 > 0 \\ \end{array}$$

Since the one-sided limits are infinite, $x = 2$ *is a vertical asymptote.*

Horizontals

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to-\infty} \frac{x^2}{x^2} = \lim_{x\to-\infty} 1 = 1$

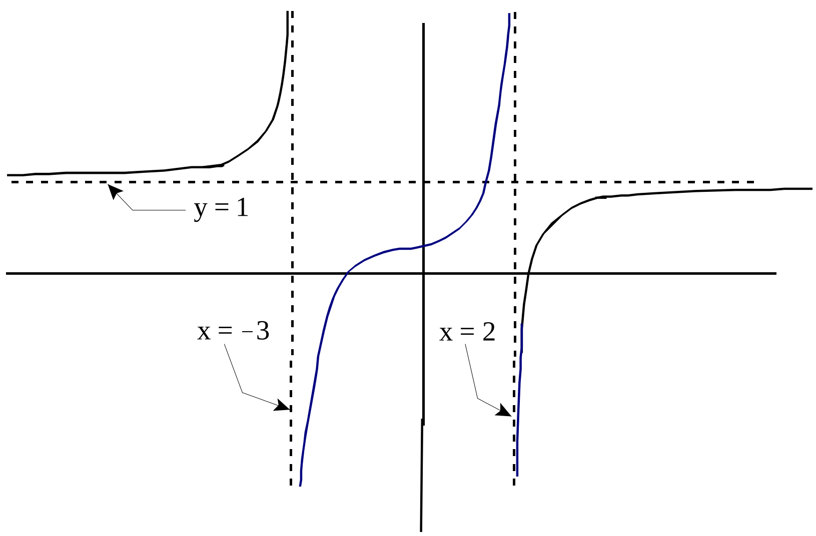
$\lim_{x\to+\infty} \frac{x^2-x-6}{x^2+x-6} = \lim_{x\to+\infty} \frac{x^2}{x^2} = \lim_{x\to+\infty} 1 = 1$

Since the limits are finite and constant, $y = 1$ is a horizontal asymptote.

Summary:

$\lim_{x\to-3^-} \frac{x^2-x-6}{x^2+x-6} = +\infty$	
$\lim_{x\to-3^+} \frac{x^2-x-6}{x^2+x-6} = -\infty$	$\lim_{x\to-\infty} \frac{x^2-x-6}{x^2+x-6} = 1$
$\lim_{x\to2^-} \frac{x^2-x-6}{x^2+x-6} = +\infty$	$\lim_{x\to+\infty} \frac{x^2-x-6}{x^2+x-6} = 1$
$\lim_{x\to2^+} \frac{x^2-x-6}{x^2+x-6} = -\infty$	

Graph $f(x) = \frac{x^2-x-6}{x^2+x-6}$



6. Compute: $\lim_{x\to 3} \frac{\sqrt{12-x}-3}{x-3} =$

Step #1 Try Plugging in:

$$\lim_{x\to 3} \frac{\sqrt{12-x}-3}{x-3} = \frac{\sqrt{12-(3)}-3}{(3)-3} = \frac{0}{0}$$

No Good -
Zero Divide!

Step #2 Try Factoring and Canceling:

$$\lim_{x\to 3} \frac{\sqrt{12-x}-3}{x-3} = \lim_{x\to 3} \frac{\sqrt{12-x}-3}{x-3} \cdot \frac{\sqrt{12-x}+3}{\sqrt{12-x}+3} = \lim_{x\to 3} \frac{\left(\sqrt{12-x}\right)^2-(3)^2}{(x-3)\left[\sqrt{12-x}+3\right]}$$

1. $= \lim_{x\to 3} \frac{(12-x)-9}{(x-3)\left[\sqrt{12-x}+3\right]} = \lim_{x\to 3} \frac{(3-x)}{(x-3)\left[\sqrt{12-x}+3\right]} = \lim_{x\to 3} \frac{-(x-3)}{(x-3)\left[\sqrt{12-x}+3\right]}$

   $= \lim_{x\to 3} \frac{-1}{\left[\sqrt{12-x}+3\right]} = \frac{-1}{\left[\sqrt{12-(3)}+3\right]} = \frac{-1}{[3+3]} = \frac{-1}{6}$

   i.e., $\lim_{x\to 3} \frac{\sqrt{12-x}-3}{x-3} = \frac{-1}{6}$

7.

$x =$	$f(x) =$
-10.1	2.5
-100.8	2.9
$-1,000.3$	2.99
$-10,000.3$	2.999
$-100,000.9$	2.9999

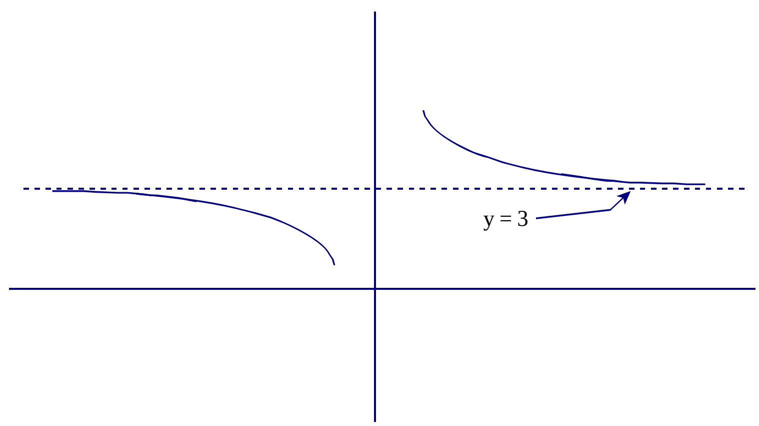
$x =$	$f(x) =$
10.1	3.5
100.8	3.1
$1,000.3$	3.01
$10,000.3$	3.001
$100,000.9$	3.0001

Based on the information in the table above, do the following:

(a) $\lim_{x\to-\infty} f(x) = 3$

(b) $\lim_{x\to+\infty} f(x) = 3$

(c) Graph $f(x)$



8. Compute: $\lim_{x\to-\infty} \frac{9x^4+4x-8x}{3x^5-8x^2-5} =$

$\lim_{x\to-\infty} \frac{9x^4+4x-8x}{3x^5-8x^2-5} = \lim_{x\to-\infty} \frac{9x^4}{3x^5} = \lim_{x\to-\infty} \frac{3}{x} = 0$

i.e., $\lim_{x\to-\infty} \frac{9x^4+4x-8x}{3x^5-8x^2-5} = 0$

Extra (5 pts - WOW!)

Compute, using the properties of limits. Document each step.

$$\lim_{x\to 1} \frac{3x^2-2x+5}{x^2-5x+3} = \underbrace{\frac{\lim_{x\to 1}(3x^2-2x+5)}{\lim_{x\to 1}(x^2-5x+3)}}_{\substack{\text{The limit of a quotient equals} \\ \text{the quotient of the limits}}} = \underbrace{\frac{\lim_{x\to 1} 3x^2 - \lim_{x\to 1} 2x + \lim_{x\to 1} 5}{\lim_{x\to 1} x^2 - \lim_{x\to 1} 5x + \lim_{x\to 1} 3}}_{\substack{\text{The limit of a sum or difference equals} \\ \text{the sum or difference of the limits}}}$$

$$= \underbrace{\frac{3\lim_{x\to 1} x^2 - 2\lim_{x\to 1} x + \lim_{x\to 1} 5}{\lim_{x\to 1} x^2 - 5\lim_{x\to 1} x + \lim_{x\to 1} 3}}_{\substack{\text{The limit of a constant times a function equals} \\ \text{the constant times the limit of the function}}} = \underbrace{\frac{3(1)^2 - 2(1) + \lim_{x\to 1} 5}{(1)^2 - 5(1) + \lim_{x\to 1} 3}}_{\substack{\lim_{x\to c} x^n = c^n \\ \lim_{x\to c} x = c}}$$

$$= \underbrace{\frac{3(1)^2 - 2(1) + 5}{(1)^2 - 5(1) + 3}}_{\substack{\text{The limit of a constant is the} \\ \text{constant itself}}} = \frac{6}{-1} = -6$$

$$\boxed{\text{i.e., } \lim_{x\to 1} \frac{\left(3x^2-2x+5\right)}{\left(x^2-5x+3\right)} = -6}$$