Laplace Transforms Homework #2

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Instructions. For problems 1 - 10, find the Laplace Transform of each function.

Note well: We will use the "linearity properties" of Laplace transforms frequently.

In particular:

1) For any canstant c, $\mathcal{L}[cf(t)] = c\mathcal{L}[f(t)]$

and

2) $\mathcal{L}[f_1(t) \pm f_2(t)] = \mathcal{L}[f_1(t)] \pm \mathcal{L}[f_2(t)]$

1. $5 - 8t^3$

$$\mathcal{L}[5-8t^3] = \mathcal{L}[5] - \mathcal{L}[8t^3] = \mathcal{L}[5] - 8\mathcal{L}[t^3] = \underbrace{\frac{5}{s}}_{\text{Transform \#1}} - 8\left[\underbrace{\frac{3!}{s^4}}_{\text{Transform \#4}}\right]$$

i.e., $\mathcal{L}[5-8t^3] = \frac{5}{s} - \frac{48}{s^4}$

2. $\frac{1}{8}\cos\left(\frac{3}{8}t\right)$

$$\mathcal{L}\left[\tfrac{1}{8}\cos\left(\tfrac{3}{8}t\right)\right] = \tfrac{1}{8}\mathcal{L}\left[\cos\left(\tfrac{3}{8}t\right)\right] = \frac{1}{8}\underbrace{\frac{s}{s^2+\left(\frac{3}{8}\right)^2}}_{\text{Transform \#9}} = \frac{s}{8\left(s^2+\frac{9}{8^2}\right)} = \frac{s}{8s^2+\frac{9}{8}} = \frac{8}{8}\frac{s}{\left(8s^2+\frac{9}{8}\right)} = \frac{8s}{64s^2+9}$$

i.e., $\mathcal{L}\left[\frac{1}{8}\cos\left(\frac{3}{8}t\right)\right] = \frac{8s}{64s^2+9}$

3. $e^{3t}\cos(2t) - e^t\sinh(5t)$

$$\mathcal{L}\left[e^{3t}\cos(2t) - e^t\sinh(5t)\right] = \mathcal{L}\left[e^{3t}\cos(2t)\right] - \mathcal{L}\left[e^t\sinh(5t)\right] = \underbrace{\frac{(s-3)}{(s-3)^2-2^2}}_{\substack{\text{Transformation \#9} \\ \text{Property \#3}}} - \underbrace{\frac{5}{(s-1)^2-5^2}}_{\substack{\text{Transformation \#12} \\ \text{Property \#3}}}$$

$$= \frac{s-3}{(s-3)^2+4} - \frac{5}{(s-1)^2-25}$$

i.e., $\mathcal{L}\left[e^{3t}\cos(2t) - e^t\sinh(5t)\right] = \frac{s-3}{(s-3)^2+4} - \frac{5}{(s-1)^2-25}$

4. $\cos(t) - \sin(t)$

$$\mathcal{L}[\cos(t) - \sin(t)] = \mathcal{L}[\cos(t)] - \mathcal{L}[\sin(t)] = \underbrace{\frac{s}{s^2+1^2}}_{\text{Transformation \#9}} - \underbrace{\frac{1}{s^2+1^2}}_{\text{Transformation \#10}}$$

$$= \frac{s-1}{s^2+1}$$

$$\boxed{\text{i.e., } \mathcal{L}[\cos(t) - \sin(t)] = \frac{s-1}{s^2+1}}$$

5. $t^7 - t^4 + 5t^2$

$$\mathcal{L}[t^7 - t^4 + 5t^2] = \mathcal{L}[t^7] - \mathcal{L}[t^4] + \mathcal{L}[5t^2] = \mathcal{L}[t^7] - \mathcal{L}[t^4] + 5\mathcal{L}[t^2]$$

$$\underbrace{\frac{7!}{s^{7+1}}}_{\text{Transformation \#5}} - \underbrace{\frac{4!}{s^{4+1}}}_{\text{Transformation \#5}} + 5\underbrace{\frac{2!}{s^{2+1}}}_{\text{Transformation \#5}} = \frac{7!}{s^8} - \frac{4!}{s^5} + \frac{5(2!)}{s^3}$$

$$\boxed{\text{i.e., } \mathcal{L}[t^7 - t^4 + 5t^2] = \frac{7!}{s^8} - \frac{4!}{s^5} + \frac{5(2!)}{s^3}}$$

6. $t\sinh(t)$

We don't have a formula on our Laplace Transforms Table that gives us $\mathcal{L}[tf(t)]$ for an arbitrary function $f(t)$. But we DO have a formula for $\mathcal{L}\left[te^{kt}\right]$ (Transform #14). Maybe we can use that along with the fact that $\sinh(t) = \frac{e^x - e^{-x}}{2}$.

$$\mathcal{L}[t\sinh(t)] = \mathcal{L}\left[t\frac{e^x-e^{-x}}{2}\right] = \mathcal{L}\left[\frac{t\left(e^x-e^{-x}\right)}{2}\right] = \mathcal{L}\left[\frac{te^x-te^{-x}}{2}\right] = \frac{1}{2}\mathcal{L}\left[\frac{1}{2}te^x - \frac{1}{2}te^{-x}\right]$$

$$= \mathcal{L}\left[\frac{1}{2}te^x\right] - \mathcal{L}\left[\frac{1}{2}te^{-x}\right] = \frac{1}{2}\mathcal{L}[te^x] - \frac{1}{2}\mathcal{L}[te^{-x}] = \underbrace{\frac{1}{2}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{2}\left(\frac{1}{(s+1)^2}\right)}_{\text{By Transform \#14}}$$

$$= \frac{1}{2}\left(\left(\frac{1}{(s-1)^2}\right) - \left(\frac{1}{(s+1)^2}\right)\right) = \frac{1}{2}\left(\left(\frac{(s+1)^2}{(s+1)^2(s-1)^2}\right) - \left(\frac{(s-1)^2}{(s+1)^2(s-1)^2}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{s^2+2s+1}{(s+1)^2(s-1)^2}\right) - \left(\frac{s^2-2s+1}{(s+1)^2(s-1)^2}\right)\right) = \frac{2s}{(s^2-1)^2}$$

$$\boxed{\text{i.e. } \mathcal{L}[t\sinh(t)] = \frac{2s}{(s^2-1)^2}}$$

7. $\frac{d}{dt}\left[te^{5t}\right]$

Hmmm . . . We **could** use the Product Rule to compute $\frac{d}{dt}\left[te^{5t}\right]$, and then compute the Laplace Transform of the result. Or we could use Transform #19: $\mathcal{L}\left[f'\left(t\right)\right] = sF\left(s\right) - f\left(0\right)$, where $f\left(t\right) = te^{5t}$.

Observe: $F\left(s\right) = \mathcal{L}\left[f\left(t\right)\right] = \underbrace{\mathcal{L}\left[te^{5t}\right] = \frac{1}{\left(s-5\right)^5}}_{\text{By Transformation \#14}}$

$\Rightarrow sF\left(s\right) = \frac{s}{\left(s-5\right)^5}$

$\mathcal{L}\left[\frac{d}{dt}\left[te^{5t}\right]\right] = sF\left(s\right) - f\left(0\right) = \frac{s}{\left(s-5\right)^5} - \left[te^{5t}\right]_{x=0} = \frac{s}{\left(s-5\right)^5} - 0 = \frac{s}{\left(s-5\right)^5}$

$\mathcal{L}\left[\frac{d}{dt}\left[te^{5t}\right]\right] = \frac{s}{\left(s-5\right)^5}$

8. $\frac{d^2}{dt^2}\left[\cos\left(t\right) + te^t\right]$

Following the same strategy that we used in the previous exercise, we refer to Transformation #20:

$\pounds\left[f''\left(t\right)\right] = s^2F\left(s\right) - sf\left(0\right) - f'\left(0\right).$

Here, $f\left(t\right) = \cos\left(t\right) + te^t$; $\quad f'\left(t\right) = -\sin\left(t\right) + \left(t+1\right)e^t$;

$f\left(0\right) = \cos\left(0\right) + \left(0\right)e^0 = 1$

$f'\left(0\right) = -\sin\left(0\right) + \left(\left(0\right)+1\right)e^0 = 1$

$F\left(s\right) = \mathcal{L}\left[f\left(t\right)\right] = \mathcal{L}\left[\cos\left(t\right) + te^t\right] = \mathcal{L}\left[\cos\left(t\right)\right] + \mathcal{L}\left[te^t\right] = \underbrace{\frac{s}{s^2+1}}_{\text{Transformation \#9}} + \underbrace{\frac{1}{\left(s-1\right)^2}}_{\text{Transformation \#14}}$

i.e., $F\left(s\right) = \frac{s}{s^2+1} + \frac{1}{\left(s-1\right)^2}$

$\pounds\left[\frac{d^2}{dt^2}\left[\cos\left(t\right) + te^t\right]\right] = \pounds\left[f''\left(t\right)\right] = s^2F\left(s\right) - sf\left(0\right) - f'\left(0\right) = s^2\left(\frac{s}{s^2+1} + \frac{1}{\left(s-1\right)^2}\right) - s\cdot 1 - 1$

$= \frac{s^3}{s^2+1^2} + \frac{s^2}{\left(s-1\right)^2} - s - 1$

i.e., $\pounds\left[\frac{d^2}{dt^2}\left[\cos\left(t\right) + te^t\right]\right] = \frac{s^3}{s^2+1^2} + \frac{s^2}{\left(s-1\right)^2} - s - 1$

9. $\int_0^t \cosh(z)\cos(t-z)\,dz$

10. $\int_0^t e^x \cos(2x)\,dx$

Let $f(t) = \int_0^t e^x \cos(2x)\,dx$

Then $f(0) = \int_0^0 e^0 \cos(2(0))\,dx = 0$

i.e., $f(0) = 0$

Also, by the Fundamental Theorem of Calculus, $f'(t) = \frac{d}{dt}[f(t)] = \frac{d}{dt}\int_0^t e^x \cos(2x)\,dx = e^t \cos(2t)$

i.e., $f'(t) = e^t \cos(2t)$

We want: $\mathcal{L}\left[\int_0^t e^x \cos(2x)\,dx\right]$

How do we compute the Laplace Transform of a definite integral???

Observe: We want: $\mathcal{L}\left[\int_0^t e^x \cos(2x)\,dx\right] = \mathcal{L}[f(t)] = F(s)$

If we find $F(s)$, we will have found $\mathcal{L}\left[\int_0^t e^x \cos(2x)\,dx\right]$

Toward this end, **Recall: Transform #19:** $\mathcal{L}[f'(t)] = sF(s) - f(0)$. In this case, $f'(t) = e^t \cos(2t)$

Thus, $\mathcal{L}[f'(t)] = \mathcal{L}[e^t \cos(2t)]$

To compute $\mathcal{L}[e^t \cos(2t)]$, first note that $\underbrace{\mathcal{L}[\cos(2t)] = \frac{s}{s^2+2^2}}_{\text{Transform \#9}} = \frac{s}{s^2+4}$.

Applying **Property #3** on **Laplace Transforms Properties** yields: $\mathcal{L}[e^t \cos(2t)] = \frac{s-1}{(s-1)^2+4}$

Summarizing what we have so far:

Given $f(t) = \int_0^t e^x \cos(2x)\,dx$, we want $F(s) = \mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^t e^x \cos(2x)\,dx\right]$

$F(s)$ can be found more easily by using the formula $\mathcal{L}[f'(t)] = sF(s) - f(0)$ and solving for $F(s)$.

By the Fundamental Theorem of Calculus, $f'(t) = \frac{d}{dt}f(t) = \frac{d}{dt}\int_0^t e^x \cos(2x)\,dx = e^t \cos(2t)$.

i.e., $f'(t) = e^t \cos(2t)$

Also: $f(0) = \int_0^0 e^x \cos(2x)\, dx = 0$

And: $\mathcal{L}[f'(t)] = \mathcal{L}[e^t \cos(2t)] = \frac{s-1}{(s-1)^2+4}$

Finally: $\mathcal{L}[f'(t)] = sF(s) - f(0) = sF(s) - 0 = sF(s)$

i.e., $\mathcal{L}[f'(t)] = \frac{s-1}{(s-1)^2+4}$ and $\mathcal{L}[f'(t)] = sF(s)$

$\Rightarrow sF(s) = \frac{s-1}{(s-1)^2+4}$

$$\boxed{\Rightarrow F(s) = \frac{s-1}{s\left[(s-1)^2+4\right]}}$$

Further Instructions For problems 11 - 20, find the Inverse Laplace Transform of each function.

Remark: *We will make copious use of two fundamental properties of Laplace Transform Inverses:*

1) $\pounds^{-1}\left[F(s) \pm G(s)\right] = \pounds^{-1}\left[F(s)\right] \pm \pounds^{-1}\left[G(s)\right]$

2) $\pounds\left[c \cdot F(s)\right] = c \cdot \pounds\left[F(s)\right]$

11. $\frac{2}{s^2+k^2}$

$\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{s^2+k^2}\right] =???$

I would like to use Formula #6 on *A Table of Laplace Transform Inverses:* $\pounds^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin(kt)$

In order to use this formula, I need to get the constant "k" in the numerator.

$\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{k}\frac{k}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{k}\frac{k}{s^2+k^2}\right] = \frac{2}{k}\pounds^{-1}\left[\frac{k}{s^2+k^2}\right] = \frac{2}{k}\sin(kt)$

i.e., $\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = \frac{2}{k}\sin(kt)$

12. $\frac{n!}{(s-k)^{n+1}}; \quad n = 1, 2, 3, \ldots$

i.e., $\pounds^{-1}\left[\frac{n!}{(s-k)^{n+1}}\right] = t^n e^{kt}$

(Using Formula #5 on *A Table of Laplace Transform Inverses)*

13. $\frac{s}{s^2-k^2}$

i.e., $\pounds^{-1}\left[\frac{s}{s^2-k^2}\right] = \cosh(kt)$

(Using Formula #10 on *A Table of Laplace Transform Inverses)*

14. $\frac{2}{(s^2+1)^2}$

This doesn't really fit any of our forms:

We have: $\frac{2ks}{(s^2+k^2)^2}$ (Formula #13) and $\frac{s^2-k^2}{(s^2+k^2)^2}$ (Formula #14), but we don't have a formula with just a plain old **constant** over $(k^2+s^2)^2$.

If we are to use either (or both) of these forms, $k=1$.

So our forms will be: $\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\sin(t)$ and $\mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = t\cos(kt)$

Q: Can we do anything with these?

Observe: $\frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$ (Using Partial Fraction Decomposition.)

$\Rightarrow \frac{2}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}$

Thus, $\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = \sin(t) - t\cos(t)$

(Using Formulas #6, 14 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = \sin(t) - t\cos(t)$

15. $\frac{s^2+3s+36}{s(s^2+13s+36)}$

$\mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right]$ =???

This doesn't even come remotely close to fitting any form given on *A Table of Laplace Transform Inverses* - We must re-express $F(s)$, using Partial Fraction Decomposition

$\frac{s^2+3s+36}{s(s^2+13s+36)} = \frac{s^2+3s+36}{s(s+9)(s+4)} = \frac{c_1}{s} + \frac{c_2}{(s+9)} + \frac{c_3}{(s+4)}$

$\Rightarrow s^2 + 3s + 36 = c_1(s+9)(s+4) + c_2 s(s+4) + c_3 s(s+9)$

$\boxed{\text{Let } s = 0}$

$\Rightarrow 36 = c_1(9)(4) = 36c_1$

$\boxed{\Rightarrow c_1 = 1}$

$\boxed{\text{Let } s = -9}$

$\Rightarrow (-9)^2 + 3(-9) + 36 = c_2(-9)((-9)+4)$

i.e. $90 = 45c_2$

$\boxed{\Rightarrow c_2 = 2}$

$\boxed{\text{Let } s = -4}$

$\Rightarrow (-4)^2 + 3(-4) + 36 = c_3(-4)((-4)+9)$

i.e. $40 = -20c_3$

$\boxed{\Rightarrow c_3 = -2}$

$$\mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{2}{(s+9)} - \frac{2}{(s+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+9)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+4)}\right]$$

$$= 1 + 2e^{-9t} - 2e^{-4t}$$

(Using Formulas #1, 4 on *A Table of Laplace Transform Inverses)*

$$\boxed{\text{i.e., } \mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right] = 1 + 2e^{-9t} - 2e^{-4t}}$$

16. $\frac{s^2+4s+36}{(s^2-4)^2}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+4s+36}{(s^2-4)^2} = \frac{1}{s+2} - \frac{1}{s-2} + \frac{3}{(s-2)^2} + \frac{2}{(s+2)^2}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+4s+36}{(s^2-4)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} - \frac{1}{s-2} + \frac{2}{(s+2)^2} + \frac{3}{(s-2)^2}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$= e^{-2t} - e^{2t} + 2te^{-2t} + 3te^{2t}$$

(Using Formulas #4, 5 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+4s+36}{(s^2-4)^2}\right] = e^{-2t} - e^{2t} + 2te^{-2t} + 3te^{2t}$

17. $\frac{s^2+2s+53}{(s+2)(s^2+49)}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+2s+53}{(s+2)(s^2+49)} = \frac{1}{s+2} + \frac{2}{s^2+49}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{2}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$

I want $\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$ to fit formula #6 on *A Table of Laplace Transform Inverses:* $\mathcal{L}^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin(kt)$

So, I will multiply the numerator by $k = 7$, and divide by $k = 7$.

$\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{2}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{7}\frac{7}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{7}\mathcal{L}^{-1}\left[\frac{7}{s^2+49}\right]$$

$$= e^{-2t} + \frac{2}{7}\sin(7t)$$

(Using Formulas #4, 6 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = e^{-2t} + \frac{2}{7}\sin(7t)$

18. $\frac{s^2+3s-18}{s(s^2-6s+9)}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+3s-18}{s(s^2-6s+9)} = \frac{(s+6)(s-3)}{s(s-3)^2} = \frac{(s+6)}{s(s-3)} = \frac{3}{s-3} - \frac{2}{s}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+3s-18}{s(s^2-6s+9)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-3} - \frac{2}{s}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s}\right]$

$= 3e^{3t} - 2$

(Using Formulas #1, 4 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+3s-18}{s(s^2-6s+9)}\right] = 3e^{3t} - 2$

19. $\frac{s^2-s+1}{s^3(s+1)}$

20. $\frac{2s-3}{(s+1)^2+16}$