Calc 2, Test 3 - SOLUTIONS

WINTER 1990

Pat Rossi Name ____________________

1. Compute: $\int \frac{x^2+6}{x^3+18x}dx =$

$u = x^3 + 18x$
$du = (3x^2 + 18)\,dx$
$\frac{1}{3}du = (x^2 + 6)\,dx$

$$\int \frac{x^2+6}{x^3+18x}dx = \int \underbrace{\frac{1}{x^3+18x}}_{\frac{1}{u}} \underbrace{\left(x^2+6\right)dx}_{\frac{1}{3}du} = \int \frac{1}{u}\cdot\frac{1}{3}du = \frac{1}{3}\int\frac{1}{u}du =$$

$$\frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3+18x\right| + C$$

2. Compute: $\frac{d}{dx}\left[e^{(\sin(x)+\cos(x))}\right] =$

$u = \sin(x) + \cos(x)$
$\frac{du}{dx} = \cos(x) - \sin(x)$

$$\underbrace{\frac{d}{dx}\left[e^{(\sin(x)+\cos(x))}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{(\sin(x)+\cos(x))}}_{e^u}\cdot\underbrace{(\cos(x)-\sin(x))}_{\frac{du}{dx}}$$

$$= e^{(\sin(x)+\cos(x))}\cdot(\cos(x)-\sin(x))$$

3. Compute: $\frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] = \frac{d}{dx}\left[\ln\left(e^{x^2}\right) + \ln(\sin(x))\right]$$

$$= \frac{d}{dx}\left[x^2 + \underbrace{\ln(\sin(x))}_{\ln(u)}\right] = 2x + \underbrace{\frac{1}{\sin x}}_{\frac{1}{u}}\cdot\underbrace{\cos(x)}_{\frac{du}{dx}} = 2x + \cot(x)$$

Alternately $\underbrace{\frac{d}{dx}\left[\ln\left(e^{x^2}\sin(x)\right)\right]}_{\frac{d}{dx}[\ln(u)]} = \underbrace{\frac{1}{e^{x^2}\sin(x)}}_{\frac{1}{u}}\cdot\underbrace{\left(\left(e^{x^2}\cdot 2x\right)\cdot(\sin(x)) + \cos(x)\cdot e^{x^2}\right)}_{\frac{du}{dx}\text{ using the product rule}} =$

$$\frac{1}{e^{x^2}\sin(x)}\cdot e^{x^2}\left(2x\cdot\sin(x)+\cos(x)\right) = \frac{2x\ \sin(x)+\cos(x)}{\sin(x)} = 2x + \cot x$$

4. Use the properties of natural logs and the facts that $\ln(2) \approx 0.7$ and $\ln(9) = 2.2$ to compute:

   (a) $\ln(4e) = \ln(4) + \ln(e) = \ln 2^2 + \ln e = 2\ln 2 + \ln e = 2(0.7) + 1 = 2.4$

   (b) $\ln(36) = \ln(36) = \ln(2^2\cdot 9) = \ln(2^2) + (9) = 2\ln 2 + \ln 9 = 2(0.7) + 2.2 = 3.6$

(c) $\ln(6) = \ln(2 \cdot 3) = \ln 2 + \ln 3 = \ln 2 + (9)^{\frac{1}{2}} = \ln 2 + \frac{1}{2}\ln 9 = 0.7 + \frac{1}{2}(2.2) = 0.7 + 1.1 = 1.8$

5. Compute: $\int_{x=1}^{x=3} \frac{e^{\frac{3}{x}}}{x^2} dx$

$u = 3x^{-1}$
$du = -3x^{-2}dx$
$-\frac{1}{3}du = x^{-2}dx$
when $x = 1;\ u = 3(1)^{-1} = 3$
when $x = 3;\ u = 3(3)^{-1} = 1$

$$\int_{x=1}^{x=3} \frac{e^{\frac{3}{x}}}{x^2} dx = \int_{x=1}^{x=3} \underbrace{e^{3x^{-1}}}_{e^u} \underbrace{x^{-2}dx}_{-\frac{1}{3}du} = \int_{u=3}^{u=1} e^u \cdot \left(-\tfrac{1}{3}du\right) = -\frac{1}{3}\int_{u=3}^{u=1} e^u du = -\frac{1}{3}\left[e^u\right]_{u=3}^{u=1} = -\frac{1}{3}\left[e^1 - e^3\right] = \frac{1}{3}\left(e^3 - e\right)$$

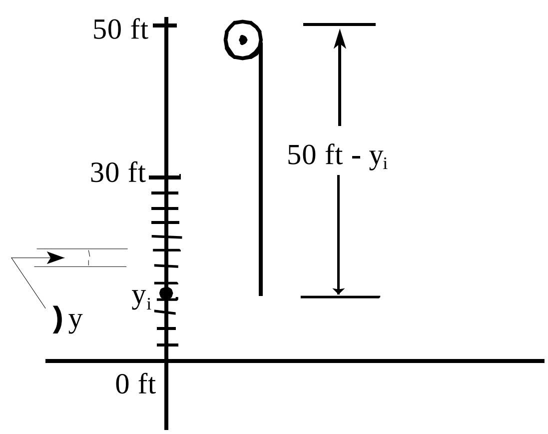
6. Find the arc length of the function $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ over the interval $[0, 1]$ on the x-axis.

$$\left.\begin{array}{|c|}\hline y' = x^{\frac{1}{2}} \\ \hline [y']^2 = x \\ \hline\end{array}\right\} \text{arc length} = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx = \int_{x=0}^{x=1} (1+x)^{\frac{1}{2}} dx$$

$u = 1 + x$
$du = dx$
when $x = 0; u = 1 + 0 = 1$
when $x = 1; u = 1 + 1 = 2$

$$\text{arc length} = \int_{x=0}^{x=1} \underbrace{(1+x)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{du} = \int_{u-1}^{u=2} u^{\frac{1}{2}} du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=1}^{u=2} = \frac{2}{3}\left[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{3}\left[2\sqrt{2} - 1\right]$$

7. A chain weighing 3lb/ft, is hanging from a winch 50 ft above the ground in such a way that the end of the chain just barely touches the ground. How much work is done winding up the chain so that the tip of the chain is 30ft above the ground?



$W_i =$ work done winding in the chain from the bottom to the top of the ith subinterval

$W_i = F_i D_i = \underbrace{(\text{length of chain}) \cdot (\text{weight per unit length of chain})}_{\text{weight of chain}} \triangle y$

$W_i = (50ft - y_i)\left(\frac{3lb}{ft}\right) \cdot \triangle y = \left(150lb - \frac{3lb}{ft} y_i\right) \triangle y$

$W_T \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \left(150lb - \frac{3lb}{ft} y_i\right) \triangle y$

Let $\Delta y \to 0$

$W_T = \int_{y=0ft}^{y=30ft} \left((150 \text{ lb}) - \frac{3lb}{ft} y\right) dy$

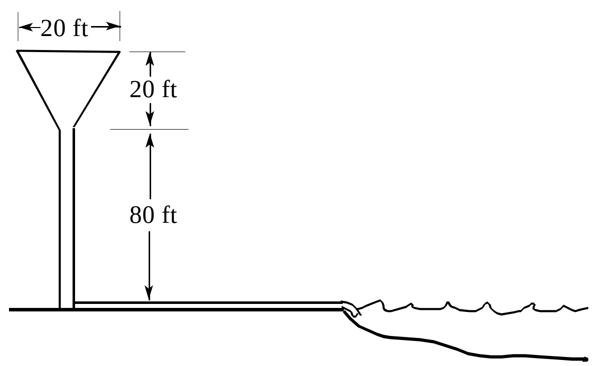
$= \left[(150 \text{ lb})\, y - \frac{3 \text{ lb}}{\text{ft}} \cdot \frac{y^2}{2}\right]_{0ft}^{30ft}$

$= \left[(150 \text{ lb})\,(30ft) - \frac{3 \text{ lb}}{\text{ft}} \cdot \frac{(30 \text{ ft})^2}{2}\right] - [0 - 0]$

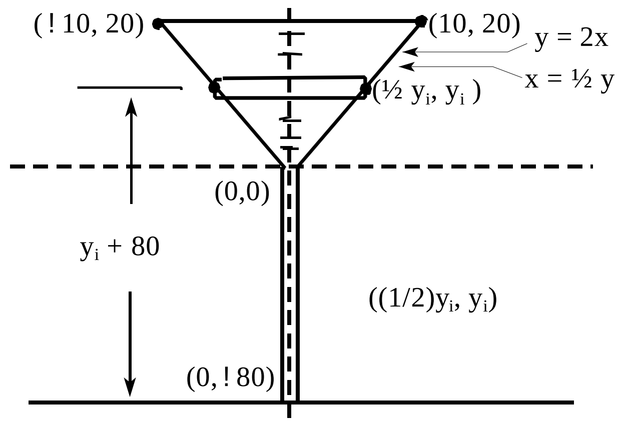
$= 4500$ ft lb-1350 ft lb

$= 3150$ ft lb

8. Water is pumped from a nearby reservoir to the tank of a water tower shown below. The tank is in the shape of an inverted cone of height 20 ft, and diameter 20 ft. If water weighs $100\frac{lb}{ft^3}$, how much work is done in pumping the tank full of water?



Assume that the tower has already been filled. We'll compute the amount of work that had to have been done in filling the tank. Slice the water in the tank into horizontal slices of width $\triangle\ y$. We'll assume that $\triangle\ y$ is small enough so that for all practical purposes, every molecule of water in the i^{th} slice is the same distance above ground.



We'll let W_i be the work done in pumping the i^{th} layer to its final height.

$W_i = F_i D_i$

$W_i = \left(\text{weight of } i^{th} \text{ layer}\right)\left(\text{Distance the } i^{th} \text{ layer is pumped}\right)$

$W_i = \left[\left(\text{volume of } i^{th} \text{ layer}\right)\left(\frac{\text{weight}}{\text{unit volume}}\right)\right](\text{Distance})$

Here, note that the i^{th} layer is disc shaped

$W_i = [(\pi r^2 \triangle y)\,\rho]\,(\text{Distance})$

$W_i = \left[\left(\pi\left(\frac{1}{2}y_i\right)^2 \triangle y\right)\rho\right](y_i + 80\text{ft})$

$W_i = \pi\rho\frac{1}{4}y_i^2\,(y_i + 80\text{ft}) \triangle y$

$W_i = \pi\rho\frac{1}{4}\,(y_i^3 + 80\text{ft }y_i^2) \triangle y$

Total Work $= W_T \approx \sum_{i=1}^n \pi\rho\frac{1}{4}\,(y_i^3 + 80\text{ft }y_i^2) \triangle y$

Let $\triangle y \to 0$

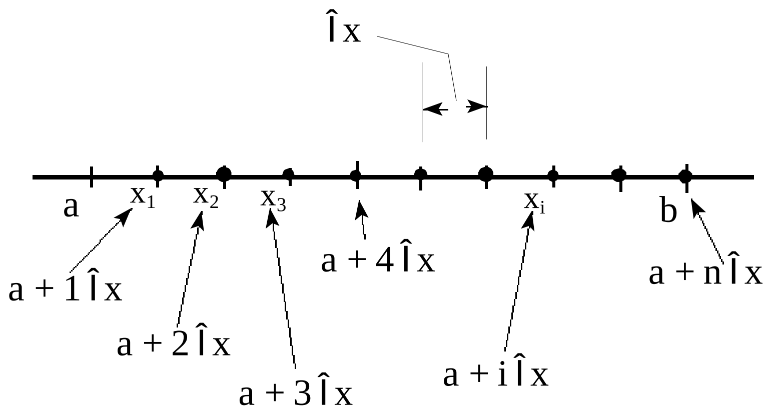
$\Rightarrow W_T = \lim_{\triangle y\to 0}\sum_{i=1}^n \pi\rho\frac{1}{4}\,(y_i^3 + 80\text{ft }y_i^2) \triangle y = \frac{\pi\rho}{4}\int_{0\text{ft}}^{20\text{ft}} (y^3 + 80\text{ft }y^2)\,dy =$
$\frac{\pi\rho}{4}\left(\frac{y^4}{4} + \frac{80\text{ft}}{3}y^3\right)\Big|_{0\text{ft}}^{20\text{ft}} = \frac{\pi\rho}{4}\left(\frac{(20\text{ft})^4}{4} + \frac{80\text{ft}}{3}(20\text{ft})^3\right) - 0 = \frac{\pi\rho}{4}\left(40000\text{ft}^4 + \frac{640000}{3}\text{ft}^4\right) =$
$\frac{\pi\rho}{4}\left(\frac{760000}{3}\text{ft}^4\right) = \pi\rho\frac{190000}{3}\text{ft}^4 = \pi\left(100\frac{\text{lb}}{\text{ft}^3}\right)\frac{190000}{3}\text{ft}^4 = \pi\frac{19{,}000{,}000}{3}$ lb-ft

9. Compute:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3 + i\frac{2}{n}\right)^2\left(\frac{2}{n}\right)$$

This fits the form of the limit of a Riemann sum. Note that since n is the number of rectangles and $\triangle x$ is the width of each rectangle, as $n \to \infty$ it follows that $\triangle x \to 0$.

Notice also that when the interval $[a, b]$ is partitioned into n subintervals of length $\triangle x$, then each subinterval has length $\triangle x = \frac{b-a}{n}$. Furthermore, note that if we let x_i be the right endpoint of the i^{th} subinterval, then $x_i = a + i \triangle x$, or $x_i = a + i\left(\frac{b-a}{n}\right)$. (See the picture below.)



Our Riemann sum can be analyzed as follows:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3+i\tfrac{2}{n}\right)^2\left(\tfrac{2}{n}\right) = \underbrace{\lim_{n\to\infty}}_{\lim_{\triangle x\to 0}} \underbrace{\sum_{i=1}^{n}}_{\sum_{i=1}^{n}} \underbrace{\left(3+i\frac{2}{n}\right)^2}_{x_i^2} \underbrace{\left(\frac{2}{n}\right)}_{\triangle x} = \underbrace{\lim_{n\to\infty}}_{\lim_{\triangle x\to 0}} \underbrace{\sum_{i=1}^{n}}_{\sum_{i=1}^{n}} \underbrace{\left(3+i\frac{2}{n}\right)^2}_{\left(a+i\left(\frac{b-a}{n}\right)\right)^2} \underbrace{\left(\frac{2}{n}\right)}_{\left(\frac{b-a}{n}\right)}$$

From this analysis, we see that $a = 3$.

Since $\frac{b-a}{n} = \frac{2}{n}$ we have that $b - a = 2$, or that $b = 2 + a = 5$.

Since our integral has the form: $\lim_{\triangle x\to 0}\sum_{i=1}^{n} x_i^2 \triangle x$ it becomes

$$\int_{a=3}^{b=5} x^2 dx = \left[\frac{x^3}{3}\right]_3^5 = \frac{5^3}{3} - \frac{3^3}{3} = \frac{98}{3}$$

10. A force of 2 Newtons is required to stretch a spring 1 meter past its point of equilibrium. (1 Newton denoted $1N$, is defined:

$$1N = \frac{1\text{kg-m}}{\text{sec}^2} \quad \text{(i.e., one kilogram meter per second squared))}$$

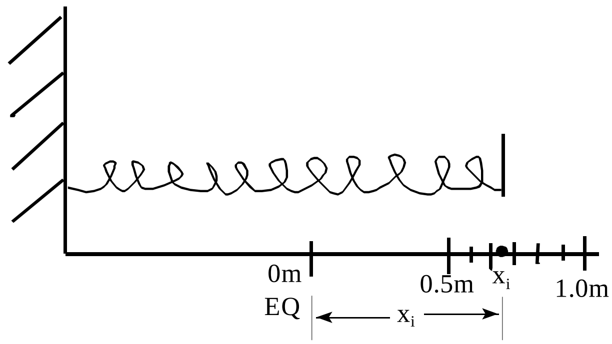
How much work is done in stretching the spring from 0.5 m past equilibrium to 1.0 m past equilibrium?

First, we need to find the "spring constant, k"

From Hooke's law, $F = ks$, where s is the distance that the spring is stretched past equilibrium, and F is the force required to stretch the spring that distance.

This means that $2N = k \cdot 1m \Rightarrow k = \frac{2N}{m}$.

Next, partition the interval over which the work is done into subintervals of length $\triangle x$, and pick a point in the i^{th} subinterval and call it x_i.



Let W_i be the work done stretching the spring from the left side to the right side of the i^{th} subinterval.

$W_i = F_i D_i$

$W_i = (ks)$ (distance over which the work is done)

$Wi = \left(\frac{2N}{m} \cdot x_i\right) \triangle x$

Total work done is given by W_T

$W_T \approx \sum_{i=1}^{n} \left(\frac{2N}{m} \cdot x_i\right) \triangle x$

$W_T = \lim_{\triangle x \to 0} \sum_{i=1}^{n} \left(\frac{2N}{m} \cdot x_i\right) \triangle x = \frac{2N}{m} \int_{x=0.5m}^{x=1.0m} x dx = \frac{2N}{m} \left[\frac{x^2}{2}\right]_{x=0.5m}^{x=1.0m} = \frac{2N}{m} \left[\frac{(1.0m)^2}{2} - \frac{(0.5m)^2}{2}\right] = \frac{2N}{m} \left(\frac{3m^2}{8}\right) = \frac{3}{4} Nm$