MTH 1125 - Test 2 (2pm Class) - Pod B - Solutions

FALL 2020

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[4x^5+5x^4+4x^3+6x^2+4x+8\sqrt{x}+10\right]=$

$\frac{d}{dx}\left[4x^5+5x^4+4x^3+6x^2+4x+8\sqrt{x}+10\right]$

$=4\left[5x^4\right]+5\left[4x^3\right]+4\left[3x^2\right]+6\left[2x\right]+4+8\left[\frac{1}{2}x^{-\frac{1}{2}}\right]+0$

$=20x^4+20x^3+12x^2+12x+4+4x^{-\frac{1}{2}}$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[4x^5+5x^4+4x^3+6x^2+4x+8\sqrt{x}+10\right]=20x^4+20x^3+12x^2+12x+4+4x^{-\frac{1}{2}}}$$

2. Compute: $\frac{d}{dx}\left[\left(3x^2+\cos(x)\right)\left(2x^3+6x\right)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(3x^2+\cos(x)\right)}_{1^{st}}\underbrace{\left(2x^3+6x\right)}_{2^{nd}}\right]=\underbrace{\left(6x-\sin(x)\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\left(2x^3+6x\right)}_{2^{nd}}+\underbrace{\left(6x^2+6\right)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(3x^2+\cos(x)\right)}_{1^{st}}$$

$$\boxed{\frac{d}{dx}\left[\left(3x^2+\cos(x)\right)\left(2x^3+6x\right)\right]=\left(6x-\sin(x)\right)\left(2x^3+6x\right)+\left(6x^2+6\right)\left(3x^2+\cos(x)\right)}$$

3. Compute: $\frac{d}{dx}\left[\frac{2x^4+2x^3+5}{4x^2+6x^2+9}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{2x^4+2x^3+5}^{\text{top}}}{\underbrace{4x^2+6x^2+9}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(8x^3+6x\right)}^{\text{top prime}}\cdot\overbrace{\left(4x^2+6x^2+9\right)}^{\text{bottom}}-\overbrace{8x+12}^{\text{bottom prime}}\cdot\overbrace{\left(2x^4+2x^3+5\right)}^{\text{top}}}{\underbrace{\left(4x^2+6x^2+9\right)^2}_{\text{bottom squared}}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\frac{2x^4+2x^3+5}{4x^2+6x^2+9}\right]=\frac{\left(8x^3+6x\right)\left(4x^2+6x^2+9\right)-(8x+12)\left(2x^4+2x^3+5\right)}{\left(4x^2+6x^2+9\right)^2}}$$

4. Compute: $\frac{d}{dx}\left[\left(4x^3+9x^2+24x\right)^6\right] =$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(4x^3+9x^2+24x\right)^6\right] = \underbrace{6\left(4x^3+9x^2+24x\right)^5}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(12x^2+18x+24\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

i.e., $\frac{d}{dx}\left[\left(4x^3+9x^2+24x\right)^6\right] = 6\left(4x^3+9x^2+24x\right)^5\left(12x^2+18x+24\right)$

5. Given that $f(x) = 2x^2 + x - 5$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(2, 5)$.

We need two things:

i. A **point** on the line (We have that: $(x_1, y_1) = (2, 5)$)

ii. The **slope** of the line (This is $f'(x_1)$)

$f'(x) = 4x + 1$

At the point $(x_1, y_1) = (2, 5)$, **the slope is** $f'(2) = 4(2) + 1 = 9$

We will use the Point-Slope equation of a line:

$(y - y_1) = m(x - x_1)$ (Where m is the slope and (x_1, y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y - 5) = 9(x - 2)$

The equation of the line tangent is $(y - 5) = 9(x - 2)$

6. Given that $w = 2x^2 + 8x$ and that $x = \tan(v)$; compute $\frac{dw}{dv}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

   We know:

   $\frac{dw}{dx} = 4x + 8$

   $\frac{dx}{dv} = \sec^2(v)$

   We want: $\frac{dw}{dv}$

   By the Liebniz form of the Chain Rule:

   $$\frac{dw}{dv} = \frac{dw}{dx}\frac{dx}{dv} = (4x+8)\sec^2(v) = \underbrace{(4\tan(v)+8)\sec^2(v)}_{\text{express solely in terms of independent variable } v}$$

i.e. $\frac{dw}{dv} = (4\tan(v)+8)\sec^2(v)$

7. Compute: $\frac{d}{dx}\left[\cot\left(4x^3+6x^2\right)\right] =$

Outer:	=	$\cot(\quad)$
Deriv. of outer	=	$-\csc^2(\quad)$

   $$\frac{d}{dx}\left[\underset{\substack{\uparrow\\ \text{outer}}}{\cot}\Big(\underset{\substack{\uparrow\\ \text{inner}}}{\underbrace{4x^3+6x^2}}\Big)\right] = \underbrace{-\csc^2\left(4x^3+6x^2\right)}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{\left(12x^2+12x\right)}_{\text{deriv. of inner}}$$

i.e., $\frac{d}{dx}\left[\cot\left(4x^3+6x^2\right)\right] = -\csc^2\left(4x^3+6x^2\right)\left(12x^2+12x\right)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^6\right] =$

$$\frac{d}{dx}\left[\underbrace{\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^6}_{(g(x))^n}\right] = \underbrace{6\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^5}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{5x^2+12x}{4x^2+8x+4}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 6\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^5 \underbrace{\frac{(10x+12)\left(4x^2+8x+4\right) - (8x+8)\left(5x^2+12x\right)}{\left(4x^2+8x+4\right)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^6\right] = 6\left(\frac{5x^2+12x}{4x^2+8x+4}\right)^5 \frac{(10x+12)\left(4x^2+8x+4\right) - (8x+8)\left(5x^2+12x\right)}{\left(4x^2+8x+4\right)^2}$

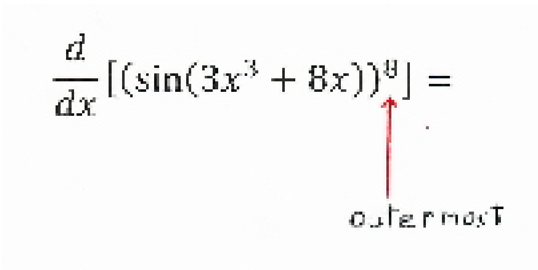
9. Compute: $\frac{d}{dx}\left[\sin^8\left(3x^3+8x\right)\right] =$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\sin\left(3x^3+8x\right)\right)^8\right]$

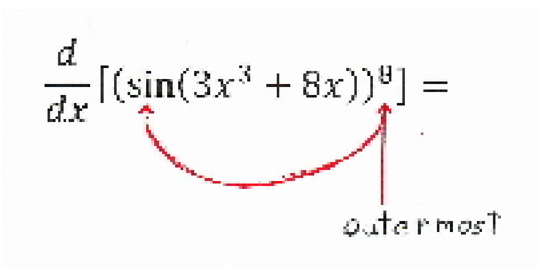
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



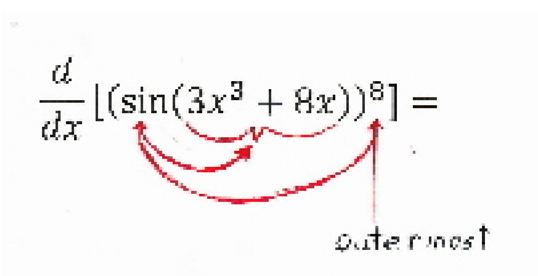
This yields: $8\left(\sin\left(3x^3+8x\right)\right)^7$

Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



This yields: $8\left(\sin\left(3x^3+8x\right)\right)^7 \cdot \cos\left(3x^3+8x\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $8\left(\sin\left(3x^3+8x\right)\right)^7 \cos\left(3x^3+8x\right)\cdot\left(9x^2+8\right)$

i.e., $\frac{d}{dx}\left[\left(\sin^8\left(3x^3+8x\right)\right)\right] = 8\left(\sin\left(3x^3+8x\right)\right)^7 \cos\left(3x^3+8x\right)\cdot\left(9x^2+8\right)$

Alternatively:

$\frac{d}{dx}\left[\left(\sin^8\left(3x^3+8x\right)\right)\right]$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE. Re-write!

$\frac{d}{dx}\left[\left(\sin\left(3x^3+8x\right)\right)^8\right]$ This is the derivative of a function, raised to a power

$$\frac{d}{dx}\left[\left(\sin\left(3x^3+8x\right)\right)^8\right] = \underbrace{8\left(\sin\left(3x^3+8x\right)\right)^7}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\sin\left(3x^3+8x\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$= 8\left(\sin\left(3x^3+8x\right)\right)^7 \cdot \underbrace{\left[\cos\left(3x^3+8x\right)\cdot\left(9x^2+8\right)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\sin^8\left(3x^3+8x\right)\right)\right] = 8\left(\sin\left(3x^3+8x\right)\right)^7\cos\left(3x^3+8x\right)\left(9x^2+8\right)$

10. Given that $x^4 + x^5 y^3 = \cos(y)$, compute y'

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[x^4 + \underbrace{x^5}_{1^{\text{st}}}\underbrace{y^3}_{2^{\text{nd}}}\right] = \frac{d}{dx}\left[\cos(y)\right]$$

$$\Rightarrow 4x^3 + \left(\underbrace{5x^4}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^3}_{2^{\text{nd}}} + \underbrace{3y^2y'}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{x^5}_{1^{\text{st}}}\right) = -\sin(y) \cdot y'$$

Simplifying, we have:

$4x^3 + 5x^4y^3 + 3x^5y^2y' = -\sin(y)\, y'$

ii. Solve algebraically for y'

a. Get y' terms on left side, all other terms on right side

$\Rightarrow 3x^5y^2y' + \sin(y)\, y' = -4x^3 - 5x^4y^3$

b. Factor out y'

$\Rightarrow y'\left(3x^5y^2 + \sin(y)\right) = -4x^3 - 5x^4y^3$

c. Divide both sides by the cofactor of y'

$y' = \frac{-4x^3 - 5x^4y^3}{3x^5y^2 + \sin(y)} = -\frac{4x^3 + 5x^4y^3}{3x^5y^2 + \sin(y)}$

$$\boxed{y' = \frac{-4x^3 - 5x^4y^3}{3x^5y^2 + \sin(y)} = -\frac{4x^3 + 5x^4y^3}{3x^5y^2 + \sin(y)}}$$

11. Given that $f(x) = 3x^2 - 5x + 2$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[3(x+\Delta x)^2 - 5(x+\Delta x) + 2\right] - \left[3x^2 - 5x + 2\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[3\left(x^2 + 2x\Delta x + \Delta x^2\right) - 5(x+\Delta x) + 2\right] - \left[3x^2 - 5x + 2\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[3x^2 + 6x\Delta x + 3\Delta x^2 - 5x - 5\Delta x + 2\right] - \left[3x^2 - 5x + 2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{6x\Delta x + 3\Delta x^2 - 5\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x(6x + 3\Delta x - 5)}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 5) = 6x + 3(0) - 5 = 6x - 5$

$$\boxed{\text{i.e., } f'(x) = 6x - 5}$$

Extra (Wow! 10 Points)

Given that $T'(x) = \frac{1}{1+x^2}$ (i.e., $\frac{d}{dx}[T(x)] = \frac{1}{1+x^2}$); compute $\frac{d}{dx}[T(\tan(x))]$

Outer:	$=$	$T(\quad)$
Deriv. of outer	$=$	$\frac{1}{1+(\quad)^2}$

$$\frac{d}{dx}\Bigg[\underset{\substack{\uparrow\\ \text{outer}}}{T}\Bigg(\underset{\substack{\uparrow\\ \text{inner}}}{\underbrace{\tan(x)}}\Bigg)\Bigg] = \underbrace{\frac{1}{1+(\tan(x))^2}}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{\sec^2(x)}_{\text{deriv. of inner}} = \frac{\sec^2(x)}{1+(\tan(x))^2} = \frac{\sec^2(x)}{\sec^2(x)} = 1$$

i.e., $\frac{d}{dx}[T(\tan(x))] = \frac{\sec^2(x)}{1+(\tan(x))^2} = 1$