MTH 1125 - Test 2 (9am Class) - Solutions

Fall 2016

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[4x^5+5x^4+10x^2+20x+2\sqrt{x}+4\right]=$

$\frac{d}{dx}\left[4x^5+5x^4+10x^2+20x+2x^{\frac{1}{2}}+4\right]$

$=4\left[5x^4\right]+5\left[4x^3\right]+10\left[2x\right]+20+2\left[\frac{1}{2}x^{-\frac{1}{2}}\right]$

$=20x^4+20x^3+20x+20+x^{-\frac{1}{2}}$

i.e., $\frac{d}{dx}\left[4x^5+5x^4+10x^2+20x+2\sqrt{x}+4\right]=20x^4+20x^3+20x+20+x^{-\frac{1}{2}}$

2. Compute: $\frac{d}{dx}\left[\sin(x)\tan(x)\right]=$

$$\frac{d}{dx}\left[\underbrace{\sin(x)}_{1^{st}}\underbrace{\tan(x)}_{2^{nd}}\right]=\underbrace{\cos(x)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\tan(x)}_{2^{nd}}+\underbrace{\sec^2(x)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\sin(x)}_{1^{st}}$$

$\frac{d}{dx}\left[\sin(x)\tan(x)\right]=\cos(x)\tan(x)+\sec^2(x)\sin(x)$

3. Compute: $\frac{d}{dx}\left[\frac{\csc(x)}{3x^2-6x+2}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{\csc(x)}^{\text{top}}}{\underbrace{3x^2-6x+2}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(-\csc(x)\cot(x)\right)}^{\text{top prime}}\cdot\overbrace{\left(3x^2-6x+2\right)}^{\text{bottom}}-\overbrace{(6x-6)}^{\text{bottom prime}}\cdot\overbrace{\csc(x)}^{\text{top}}}{\underbrace{\left(3x^2-6x+2\right)^2}_{\text{bottom squared}}}$$

i.e., $\frac{d}{dx}\left[\frac{\csc(x)}{3x^2-6x+2}\right]=\frac{-\csc(x)\cot(x)\left(3x^2-6x+2\right)-(6x-6)\csc(x)}{\left(3x^2-6x+2\right)^2}$

4. Compute: $\frac{d}{dx}\left[\left(6x^5+8x^3\right)^{20}\right]=$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(6x^5+8x^3\right)^{20}\right]=\underbrace{20\left(6x^5+8x^3\right)^{19}}_{\substack{\text{power rule}\\ \text{as usual}}}\cdot\underbrace{\left(30x^4+24x^2\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

i.e., $\frac{d}{dx}\left[\left(6x^5+8x^3\right)^{20}\right]=20\left(6x^5+8x^3\right)^{19}\left(30x^4+24x^2\right)$

5. Given that $f(x)=x^2-2x+1$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(2,1)$.

   We need two things:

   i. A point on the line (We have that: $(x_1,y_1)=(2,1)$)

   ii. The slope of the line (This is $f'(x_1)$)

   $f'(x)=2x-2$

   At the point $(x_1,y_1)=(2,1)$, **the slope is** $f'(1)=2(2)-2=2$

   We will use the Point-Slope equation of a line:

   $y-y_1=m(x-x_1)$ (Where m is the slope and (x_1,y_1) is a known point on the line.)

   Thus, the equation of the line tangent to the graph of $f(x)$ is:

   $y-1=2(x-2)$

The equation of the line tangent is $y-1=2(x-2)$

6. Given that $y = 3x^2 + 3x$ and that $x = \sin(t)$; compute $\frac{dy}{dt}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

   We know:

   $\frac{dy}{dx} = 6x + 3$

   $\frac{dx}{dt} = \cos(t)$

   We want: $\frac{dy}{dt}$

   By the Liebniz form of the Chain Rule:

   $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (6x+3)\cos(t) = \underbrace{(6\sin(t)+3)\cos(t)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } t}}$$

i.e. $\frac{dy}{dt} = (6\sin(t)+3)\cos(t)$

7. Compute: $\frac{d}{dx}\left[\tan\left(2x^2+4x+3\right)\right] =$

Outer:	$=$	$\tan(\quad)$
Deriv. of outer	$=$	$\sec^2(\quad)$

   $$\frac{d}{dx}\left[\begin{array}{ll} \tan\Big(\underbrace{2x^2+4x+3}\Big) & \\ \uparrow \qquad\qquad \uparrow & \\ \text{outer} \quad \text{inner} & \end{array}\right] = \underbrace{\sec^2\left(2x^2+4x+3\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{(4x+4)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\tan\left(2x^2+4x+3\right)\right] = \sec^2\left(2x^2+4x+3\right)(4x+4)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{8x^5+8x}{3x^2+9x}\right)^5\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{8x^5+8x}{3x^2+9x}\right)^5}_{(g(x))^n}\right] = \underbrace{5\left(\frac{8x^5+8x}{3x^2+9x}\right)^4}_{\substack{\text{power rule}\\\text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{8x^5+8x}{3x^2+9x}\right]\right)}_{\substack{\text{deriv of}\\\text{inner Function}}}$$

$$= 5\left(\frac{8x^5+8x}{3x^2+9x}\right)^4 \underbrace{\frac{(40x^4+8)(3x^2+9x) - (6x+9)(8x^5+8x)}{(3x^2+9x)^2}}_{\substack{\text{quotient}\\\text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{8x^5+8x}{3x^2+9x}\right)^5\right] = 5\left(\frac{8x^5+8x}{3x^2+9x}\right)^4 \cdot \frac{\left(40x^4+8\right)\left(3x^2+9x\right) - (6x+9)\left(8x^5+8x\right)}{(3x^2+9x)^2}$

9. Compute: $\frac{d}{dx}\left[\sin^5(5x^2+10x)\right] =$ Re-write!

$\frac{d}{dx}\left[(\sin(5x^2+10x))^5\right]$ This is the derivative of a function, raised to a power

$$\frac{d}{dx}\left[(\sin(5x^2+10x))^5\right] = \underbrace{5\left(\sin\left(5x^2+10x\right)\right)^4}_{\substack{\text{power rule}\\\text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\sin\left(5x^2+10x\right)\right]\right)}_{\substack{\text{derivative}\\\text{of inner}}}$$

$$= 5\left(\sin(5x^2+10x)\right)^4 \cdot \underbrace{\left(\cos\left(5x^2+10x\right)\right) \cdot (10x+10)}_{\substack{\text{Chain}\\\text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\sin^5(5x^2+10x)\right] = 5\left(\sin(5x^2+10x)\right)^4\left(\cos(5x^2+10x)\right) \cdot (10x+10)$

10. Given that $L'(x) = \frac{1}{x}$; compute $\frac{d}{dx}[L(\tan(x))]$

$$\begin{array}{|rcl|} \text{Outer:} & = & L(\quad) \\ \text{Deriv. of outer} & = & \frac{1}{(\quad)} \end{array}$$

$$\frac{d}{dx}\left[\underset{\substack{\uparrow \\ \text{outer}}}{L}\left(\underset{\substack{\uparrow \\ \text{inner}}}{\underbrace{\tan(x)}}\right)\right] = \underbrace{\frac{1}{(\tan(x))}}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\sec^2(x)}_{\substack{\text{deriv. of} \\ \text{inner}}} = \frac{\sec^2(x)}{\tan(x)}$$

$$\boxed{\text{i.e., } \frac{d}{dx}[L(\tan(x))] = \frac{1}{\tan(x)} \cdot \sec^2(x) = \frac{\sec^2(x)}{\tan(x)}}$$

11. Given that $f(x) = x^2 - 4x + 2$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2-4(x+\Delta x)+2\right]-\left[x^2-4x+2\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[\left(x^2+2x\Delta x+\Delta x^2\right)-4(x+\Delta x)+2\right]-\left[x^2-4x+2\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[x^2+2x\Delta x+\Delta x^2-4x-4\Delta x+2\right]-\left[x^2-4x+2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x+\Delta x^2-4\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x-4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x + (0) - 4 = 2x - 4$

$$\boxed{\text{i.e., } f'(x) = 2x - 4}$$

12. Given that $x^3 + 5y^4 = 3x^2y^3$; compute y'

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[x^3 + 5y^4\right] = \frac{d}{dx}\left[\underbrace{3x^2}_{1^{\text{st}}}\underbrace{y^3}_{2^{\text{nd}}}\right]$$

$$\Rightarrow 3x^2 + 20y^3 \cdot y' = \underbrace{6x}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^3}_{2^{\text{nd}}} + \underbrace{3y^2 \cdot y'}_{2^{\text{nd}}\text{prime}} \cdot \underbrace{3x^2}_{1^{\text{st}}}$$

ii. Solve algebraically for y'

a. Get y' terms on left side, all other terms on right side

$\Rightarrow 20y^3y' - 3y^2y'3x^2 = 6xy^3 - 3x^2$

b. Factor out y'

$\Rightarrow (20y^3 - 3y^2 3x^2)\, y' = 6xy^3 - 3x^2$

c. Divide both sides by the cofactor of y'

$y' = \frac{6xy^3-3x^2}{20y^3-3y^2 3x^2} = \frac{6xy^3-3x^2}{20y^3-9y^2x^2}$

$$\boxed{y' = \frac{6xy^3-3x^2}{20y^3-3y^2 3x^2} = \frac{6xy^3-3x^2}{20y^3-9y^2x^2}}$$