MTH 1126 - Test #4 - Solutions

SPRING 2024 - 9AM CLASS

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Show CLEARLY how you arrive at your answers.

In Exercises 1-2, Determine convergence/divergence. If the integral converges, find its value.

1. $\int_5^\infty \frac{1}{(x-1)^{\frac{3}{2}}}dx =$

$$\begin{array}{|lcl|} \hline u & = & x-1 \\ \frac{du}{dx} & = & 1 \\ du & = & dx \\ \hline \end{array}$$

$$\begin{array}{|l|} \hline \text{When } x=5;\ u=x-1=4 \\ \text{When } x=\infty;\ u=x-1=\infty \\ \hline \end{array}$$

$$\int\limits_{x=5}^{x=\infty} \frac{1}{(x-1)^{\frac{3}{2}}}dx = \int\limits_{u=4}^{u=\infty} \frac{1}{u^{\frac{3}{2}}}du = \lim_{b\to\infty} \int\limits_{u=4}^{u=b} u^{-\frac{3}{2}}du = \lim_{b\to\infty}\left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_{u=4}^{u=b} = \lim_{b\to\infty}\left[-\frac{2}{u^{\frac{1}{2}}}\right]_{u=4}^{u=b}$$

$$= \lim_{b\to\infty}\left[-\frac{2}{b^{\frac{1}{2}}} - \left(-\frac{2}{4^{\frac{1}{2}}}\right)\right] = 0-(-1) = 1$$

i.e., $\int_5^\infty \frac{1}{(x-1)^{\frac{3}{2}}}dx = 1$ (Integral **Converges)**

2. $\int_5^{14} \frac{1}{(x-5)^{\frac{1}{2}}} dx =$

(Because $\frac{1}{(x-5)^{\frac{1}{2}}}$ is discontinuous at $x = 5$, this is an improper integral.)

$$\begin{array}{|lcl|} \hline u & = & x-5 \\ \frac{du}{dx} & = & 1 \\ du & = & dx \\ \hline \end{array}$$

$$\begin{array}{|l|} \hline \text{When } x = 5;\ u = x-5 = 0 \\ \text{When } x = 14;\ u = x-5 = 9 \\ \hline \end{array}$$

$$\int_{x=5}^{x=14} \frac{1}{(x-5)^{\frac{1}{2}}} dx = \int_{u=0}^{u=9} \frac{1}{u^{\frac{1}{2}}} du = \int_{u=0}^{u=9} u^{-\frac{1}{2}} du = \lim_{a \to 0^+} \int_{u=a}^{u=9} u^{-\frac{1}{2}} du = \lim_{a \to 0^+} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{u=a}^{u=9} = \lim_{a \to 0^+} \left[2u^{\frac{1}{2}} \right]_{u=a}^{u=9}$$

$$= \lim_{a \to 0^+} \left[2(9)^{\frac{1}{2}} - 2(a)^{\frac{1}{2}} \right] = [2(3) - 2(0)] = 6$$

> i.e. $\int_5^{14} \frac{1}{(x-5)^{\frac{1}{2}}} dx = 6$ (Integral **Converges**)

3. Determine convergence/divergence of the sequence whose n^{th} term is given by:

$$a_n = \frac{2n}{n+1}.$$

(i.e., Determine convergence/divergence of the sequence:

$$\left\{ \frac{2n}{n+1} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{12}{7}, \ldots, \frac{2n}{n+1}, \ldots \right\}.)$$

Observe: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2n}{n} = \lim_{n\to\infty} 2 = 2$

Since $\lim_{n\to\infty} a_n$ is a finite real number, the sequence converges to that limit.

> $\left\{ \frac{2n}{n+1} \right\}_{n=1}^{\infty}$ **converges** to 2.

4. Determine convergence/divergence of the given series. (Justify your answer!) **If the series converges, determine its sum.**

$$\sum_{n=3}^{\infty} \frac{4}{n^2-4} = \frac{4}{5} + \frac{4}{12} + \frac{4}{21} + \dots$$

"If the series converges, determine its sum." In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping (Collapsing) Sum."

The series $\sum_{n=3}^{\infty} \frac{4}{n^2-4}$ is definitely NOT Geometric.

Maybe it can be written as a "Telescoping (Collapsing) Sum."

So let's see if we can express $a_n = \frac{4}{n^2-4}$ as the difference of two terms.

$$\frac{4}{n^2-4} = \frac{4}{(n-2)(n+2)} = \frac{C_1}{n-2} + \frac{C_2}{n+2}$$

i.e., $\frac{4}{(n-2)(n+2)} = \frac{C_1}{n-2} + \frac{C_2}{n+2}$

$$\Rightarrow \frac{4}{(n-2)(n+2)}(n-2)(n+2) = \frac{C_1}{n-2}(n-2)(n+2) + \frac{C_2}{n+2}(n-2)(n+2)$$

$$\Rightarrow 4 = C_1(n+2) + C_2(n-2)$$

$\boxed{n=2}$ $\Rightarrow 4 = C_1(4)$

$\boxed{\Rightarrow C_1 = 1}$

$\boxed{n=-2}$ $\Rightarrow 4 = C_2(-4)$

$\boxed{\Rightarrow C_2 = -1}$

Thus, $\frac{4}{n^2-4} = \frac{1}{n-2} - \frac{1}{n+2}$

$$\Rightarrow \sum_{n=3}^{N} \frac{4}{n^2-4} = \sum_{n=3}^{N}\left(\frac{1}{n-2} - \frac{1}{n+2}\right) = \left(1-\frac{1}{5}\right) + \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \left(\frac{1}{5}-\frac{1}{9}\right)$$

$$+ \left(\frac{1}{6}-\frac{1}{10}\right) + \left(\frac{1}{7}-\frac{1}{11}\right) + \dots + \left(\frac{1}{N-7}-\frac{1}{N-3}\right) + \left(\frac{1}{N-6}-\frac{1}{N-2}\right) + \left(\frac{1}{N-5}-\frac{1}{N-1}\right)$$

$$+ \left(\frac{1}{N-4}-\frac{1}{N}\right) + \left(\frac{1}{N-3}-\frac{1}{N+1}\right) + \left(\frac{1}{N-2}-\frac{1}{N+2}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots - \frac{1}{N-1} - \frac{1}{N} - \frac{1}{N+1} - \frac{1}{N+2}$$

i.e. $\sum_{n=3}^{N}\left(\frac{1}{n-2} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots - \frac{1}{N-1} - \frac{1}{N} - \frac{1}{N+1} - \frac{1}{N+2}$

Consequently:

$$\sum_{n=3}^{\infty} \frac{4}{n^2-4} = \lim_{N\to\infty} \sum_{n=3}^{N} \left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

$$= \lim_{N\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots - \frac{1}{N-1} - \frac{1}{N} - \frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{25}{12}$$

i.e., $\sum_{n=3}^{\infty} \frac{4}{n^2-4}$ **converges** and is equal to $\frac{25}{12}$

In Exercises 5-6, determine convergence/divergence of the given series. (Justify your answers!) **If the series converges, determine its sum.**

5. $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots + \left(\frac{2}{5}\right)^n + \ldots$

   "If the series converges, determine its sum." In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   Notice that each term after the first term is equal to $\frac{4}{5}$ times its predecessor.

   The series is geometric with ratio $r = \frac{2}{5}$

   Since $|r| < 1$, the series converges to $\frac{1^{\text{st}} \text{ term}}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$

   The series **converges** to $\frac{5}{3}$

6. $\sum_{n=1}^{\infty} \frac{n}{5n+1} =$

   First, note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{5n+1} = \frac{1}{5}$

   Since $\lim_{n\to\infty} a_n \neq 0$, the series **diverges.**

   i.e., $\sum_{n=1}^{\infty} \frac{n}{5n+1}$ **diverges** by the "n^{th} term Test."

In Exercises 7-9, determine convergence/divergence of the given series. (Justify your answers!)

7. $\sum_{n=1}^{\infty} \frac{1}{3n^2-1}$

There are a few different ways that we can try to do this.

We can compare $\sum_{n=1}^{\infty} \frac{1}{3n^2-1}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series with $p = 2 > 1$.

Since $\underbrace{\frac{1}{3n^2-1}}_{a_n} < \underbrace{\frac{1}{n^2}}_{b_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3n^2-1}$ converges by the **Direct Comparison Test.**

(i.e. the fact that the "larger series" converges implies that the "smaller series" converges also.)

i.e., $\sum_{n=4}^{\infty} \frac{1}{3n^2-1}$ **diverges** by the **Direct Comparison** with $\sum_{n=4}^{\infty} \frac{1}{n^2}$

Alternatively, we can use the **Limit** Comarison Test.

Observe: $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{3n^2-1}\right)}{\left(\frac{1}{n^2}\right)}\right| = \lim_{n\to\infty} \frac{n^2}{3n^2-1} = \lim_{n\to\infty} \frac{n^2}{3n^2} = \frac{1}{3}$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=4}^{\infty} \frac{1}{n^2}$, is a convergent p-series (with $p = 2$), $\sum_{n=4}^{\infty} \frac{1}{3n^2-1}$ converges also, by the **Limit Comparison Test.**

i.e., $\sum_{n=4}^{\infty} \frac{1}{3n^2-1}$ **diverges** by the **Limit Comparison Test** with $\sum_{n=4}^{\infty} \frac{1}{n^2}$

8. $\sum_{n=1}^{\infty} \frac{1}{n+2}$

There may be a few ways to do this.

First, we can compare $\sum_{n=1}^{\infty} \frac{1}{n+2}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent Harmonic Series.

Since $\underbrace{\frac{1}{n+2}}_{a_n} < \underbrace{\frac{1}{n}}_{b_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we can conclude nothing from the Direct Comparison Test.

(i.e., since the "larger series" diverges, this tells us nothing about the "smaller series.")

Alternatively: Applying the Limit Comparison Test, we have:

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n+2}\right)}{\left(\frac{1}{n}\right)}\right| = \lim_{n\to\infty} \frac{n}{n+2} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=1}^{\infty} \frac{1}{n}$, is the divergent Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n+2}$ diverges also, by the **Limit Comparison Test.**

Alternatively: $\int_1^{\infty} \frac{1}{n+2} dn = \lim_{b\to\infty} \int_1^b \underbrace{\frac{1}{n+2}}_{\frac{1}{u}} \underbrace{dn}_{du} = \lim_{b\to\infty} \left[\ln(n+2)\right]_1^b$

$$= \lim_{b\to\infty} \left[\ln(b+2) - \ln(1+2)\right] = \infty$$

$\sum_{n=1}^{\infty} \frac{1}{n+2}$ **diverges** by the **Integral Test**

Alternatively: $\sum_{n=1}^{\infty} \frac{1}{n+2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$ is the Harmonic Series with the first two terms deleted. Since the Harmonic Series diverges, **this series diverges** also, because adding or deleting finitely many terms from a series does not change whether the series converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{n+2}$ **diverges** by **Limit Comparison** with $\sum_{n=2}^{\infty} \frac{1}{n}$
Or $\sum_{n=1}^{\infty} \frac{1}{n+2}$ **diverges** by the **Integral Test.**
Or, $\sum_{n=1}^{\infty} \frac{1}{n+2}$ **diverges** because it is the Harmonic Series with finitely many terms deleted.

9. Determine convergence/divergence of the given series. (Justify your answer!)

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots$

Observe: [1] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$

Also: $\frac{1}{2n+1} > \frac{1}{2(n+1)+1}$ i.e. $a_n > a_{n+1}$

Finally: the series is alternating.

By the Alternating Series Test, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$ **converges**

10. Determine convergence/divergence of the given series. (Justify your answer!)

$\sum_{n=1}^{\infty} \left(\frac{n+3}{5n+1}\right)^n$

The n^{th} term, a_n is something **raised to the** n^{th} **power,** so this series is a good candidate for the n^{th} **Root Test.**

Observe: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+3}{5n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{n+3}{5n+1}\right) = \lim_{n\to\infty} \left(\frac{n}{5n}\right)$

$$= \lim_{n\to\infty} \left(\frac{1}{5}\right) = \frac{1}{5}$$

i.e., $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{5}$

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series **converges.** by the n^{th} **Root Test.**

$\sum_{n=1}^{\infty} \left(\frac{n+3}{5n+1}\right)^n$ **converges** by the n^{th} **Root Test.**

11. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

The n^{th} term a_n contains a **factorial,** so this is a good candidate for the **Ratio Test.**

Observe: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\left(\frac{2^n}{n!}\right)}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series **converges.**

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ **converges** by the **Ratio Test.**