MTH 1125 Test #1 - Solutions

Summer 2008

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} =$

   1. Try Plugging in:

      $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} = \frac{(2)^2-3(2)+2}{(2)^2+(2)-6} = \frac{0}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

      $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{(x-1)}{(x+3)} = \frac{(2)-1}{(2)+3} = \frac{1}{5}$

      i.e., $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} = \frac{1}{5}$

2. Compute: $\lim_{x\to 2} \frac{x^3-x-2}{x^2+x+2} =$

   1. Try Plugging in:

      $\lim_{x\to 2} \frac{x^3-x-2}{x^2+x+2} = \frac{(2)^3-(2)-2}{(2)^2+(2)+2} = \frac{4}{8} = \frac{1}{2}$

      i.e., $\lim_{x\to 2} \frac{x^3-x-2}{x^2+x+2} = \frac{1}{2}$

3. Compute: $\lim_{x\to 2} \frac{x+5}{x^2-x-2} =$

   1. Try Plugging in:

      $\lim_{x\to 2} \frac{x+5}{x^2-x-2} = \frac{(2)+5}{(2)^2-(2)-2} = \frac{7}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

      No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

   3. Analyze the one-sided limits:

      $\lim_{x\to 2^-} \frac{x+5}{x^2-x-2} = \lim_{x\to 2^-} \frac{x+5}{(x+1)(x-2)} = \frac{7}{(3)(-\varepsilon)} = \frac{\left(\frac{7}{3}\right)}{(-\varepsilon)} = -\infty$

      $x \to 2^-$
      $\Rightarrow x < 2$
      $\Rightarrow x - 2 < 0$

$$\lim_{x\to 2^+} \frac{x+5}{x^2-x-2} = \lim_{x\to 2^+} \frac{x+5}{(x+1)(x-2)} = \frac{7}{(3)(\varepsilon)} = \frac{\left(\frac{7}{3}\right)}{(\varepsilon)} = +\infty$$

$$\begin{array}{ll} & x \to 2^+ \\ \Rightarrow & x > 2 \\ \Rightarrow & x - 2 > 0 \end{array}$$

Since the one-sided limits are not equal, $\lim_{x\to 2} \frac{x+5}{x^2-x-2}$ **Does Not Exist!**

4. Compute: $\lim_{x\to\infty} \frac{3x^5-4x^3+5x^2+x+1}{4x^7+x^2} =$

$$\lim_{x\to\infty} \frac{3x^5-4x^3+5x^2+x+1}{4x^7+x^2} = \lim_{x\to\infty} \frac{3x^5}{4x^7} = \lim_{x\to\infty} \frac{3}{4x^2} = 0$$

i.e., $\lim_{x\to\infty} \frac{3x^5-4x^3+5x^2+x+1}{4x^7+x^2} = 0$

5. $f(x) = 8x^5 + 4x^3 + 2x^2 + 4x + 5;$ Compute: $f'(x)$.

$$f'(x) = 8(5x^4) + 4(3x^2) + 2(2x^1) + 4(1) + 0 = 40x^4 + 12x^2 + 4x + 4$$

i.e., $f'(x) = 40x^4 + 12x^2 + 4x + 4$

6. $\frac{d}{dx}[3\sin(x) + 2\cos(x)] =$

$$\frac{d}{dx}[3\sin(x) + 2\cos(x)] = 3(\cos(x)) + 2(-\sin(x)) = 3\cos(x) - 2\sin(x)$$

i.e., $\frac{d}{dx}[3\sin(x) + 2\cos(x)] = 3\cos(x) - 2\sin(x)$

7. Find the asymptotes and graph: $f(x) = \frac{4x^2-1}{x^2+x-2}$

Verticals

1. Find x-values that cause division by zero.

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x+2)(x-1) = 0$

$\Rightarrow x = -2$ and $x = 1$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$\lim_{x \to -2^-} \frac{4x^2-1}{x^2+x-2} = \lim_{x \to -2^-} \frac{4x^2-1}{(x+2)(x-1)} = \frac{15}{(-\varepsilon)(-3)} = \frac{15}{(\varepsilon)(3)} = \frac{5}{\varepsilon} = +\infty$

$$\begin{array}{ll} & x \to -2^- \\ \Rightarrow & x < -2 \\ \Rightarrow & x + 2 < 0 \end{array}$$

$\lim_{x \to -2^+} \frac{4x^2-1}{x^2+x-2} = \lim_{x \to -2^+} \frac{4x^2-1}{(x+2)(x-1)} = \frac{15}{(\varepsilon)(-3)} = \frac{-5}{\varepsilon} = -\infty$

$$\begin{array}{ll} & x \to -2^+ \\ \Rightarrow & x > -2 \\ \Rightarrow & x + 2 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = -2$ *is a vertical asymptote.*

$\lim_{x \to 1^-} \frac{4x^2-1}{x^2+x-2} = \lim_{x \to 1^-} \frac{4x^2-1}{(x+2)(x-1)} = \frac{3}{(3)(-\varepsilon)} = \frac{1}{(-\varepsilon)} = -\infty$

$$\begin{array}{ll} & x \to 1^- \\ \Rightarrow & x < 1 \\ \Rightarrow & x - 1 < 0 \end{array}$$

$\lim_{x\to 1^+} \frac{4x^2-1}{x^2+x-2} = \lim_{x\to 1^+} \frac{4x^2-1}{(x+2)(x-1)} = \frac{3}{(3)(\varepsilon)} = \frac{1}{(\varepsilon)} = +\infty$

$$\begin{array}{ll} & x \to 1^+ \\ \Rightarrow & x > 1 \\ \Rightarrow & x - 1 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = 1$ *is a vertical asymptote.*

Horizontals

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to -\infty} \frac{4x^2-1}{x^2+x-2} = \lim_{x\to -\infty} \frac{4x^2}{x^2} = \lim_{x\to -\infty} 4 = 4$

$\lim_{x\to +\infty} \frac{4x^2-1}{x^2+x-2} = \lim_{x\to +\infty} \frac{4x^2}{x^2} = \lim_{x\to +\infty} 4 = 4$

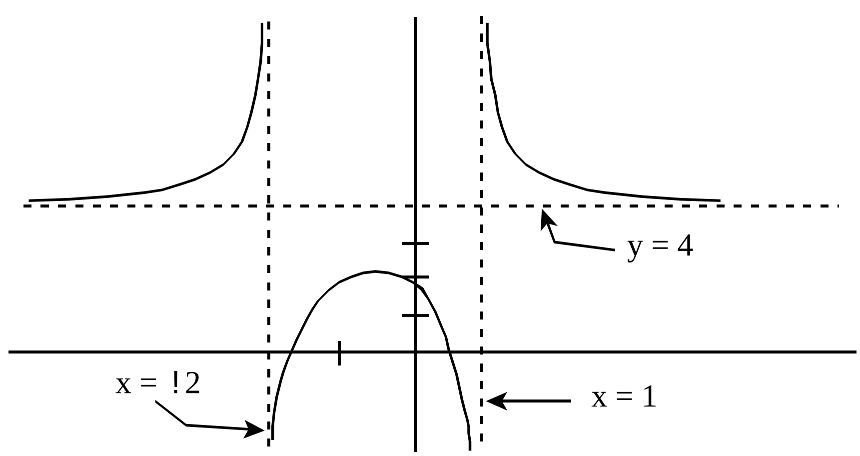
Since the limits are finite and constant, $y = 4$ is a horizontal asymptote.

Summary:

$\lim_{x\to -2^-} \frac{4x^2-1}{x^2+x-2} = +\infty$
$\lim_{x\to -2^+} \frac{4x^2-1}{x^2+x-2} = -\infty$
$\lim_{x\to 1^-} \frac{4x^2-1}{x^2+x-2} = -\infty$
$\lim_{x\to 1^+} \frac{4x^2-1}{x^2+x-2} = +\infty$

$\lim_{x\to -\infty} \frac{4x^2-1}{x^2+x-2} = 4$
$\lim_{x\to +\infty} \frac{4x^2-1}{x^2+x-2} = 4$

Graph $f(x) = \frac{4x^2-1}{x^2+x-2}$



8.

$x =$	$f(x) =$
2.5	15.1
2.9	227.8
2.99	1212.3
2.999	21156.3
2.9999	834561.9

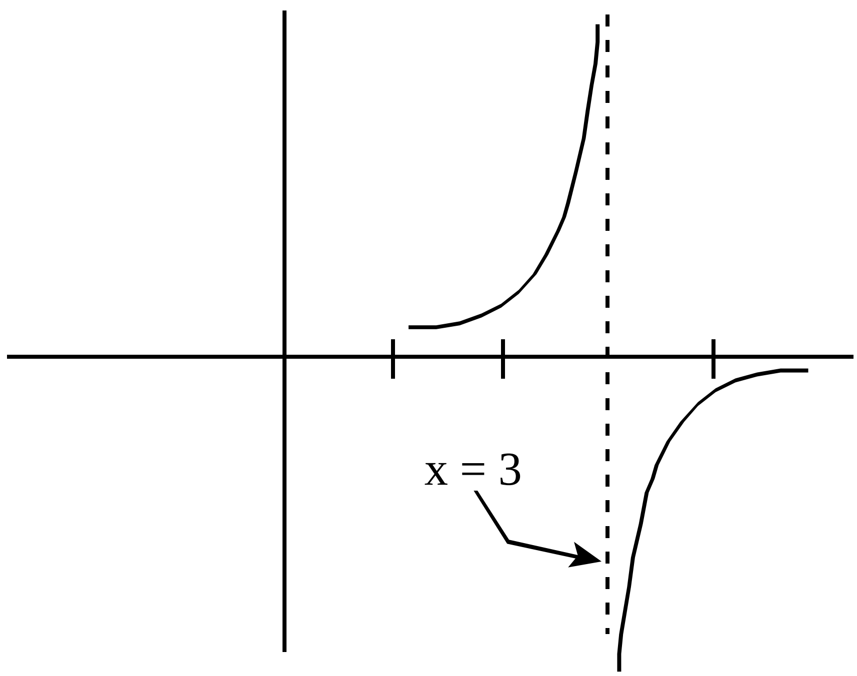
$x =$	$f(x) =$
3.5	-15.1
3.1	-227.8
3.01	-1212.3
3.001	-21156.3
3.0001	-834561.9

Based on the information in the table above, do the following:

(a) $\lim_{x \to 3^-} f(x) = +\infty$

(b) $\lim_{x \to 3^+} f(x) = -\infty$

(c) Graph $f(x)$



9. Determine whether or not $f(x)$ is continuous at the point $x = 2$.

$$f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{for } x \neq 2 \\ \\ 12 & \text{for } x = 2 \end{cases}$$

Recall: $f(x)$ is continuous at the point $x = 2$ exactly when $\lim_{x\to 2} f(x) = f(2)$

Observe:

$$\begin{aligned} \lim_{x\to 2} f(x) &= \lim_{x\to 2} \tfrac{x^3-8}{x-2} = \lim_{x\to 2} \tfrac{(x-2)\left(x^2+2x+4\right)}{(x-2)} = \lim_{x\to 2} \left(x^2+2x+4\right) \\ &= \left((2)^2 + 2(2) + 4\right) = 12 \end{aligned}$$

Note that $\lim_{x\to 2} f(x) = 12 = f(2)$.

Since $\lim_{x\to 2} f(x) = f(2)$, $f(x)$ is continuous at $x = 2$.

10. $f(x) = 2x^2 + 3x$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left(2(x+\Delta x)^2+3(x+\Delta x)\right)-\left(2x^2+3x\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left(2\left(x^2+2x\Delta x+\Delta x^2\right)+3(x+\Delta x)\right)-\left(2x^2+3x\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x^2+4x\Delta x+2\Delta x^2+3x+3\Delta x-\left(2x^2+3x\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{4x\Delta x+2\Delta x^2+3\Delta x}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(4x+2\Delta x+3)}{\Delta x} = \lim_{\Delta x\to 0} (4x + 2\Delta x + 3)$

$= 4x + 2(0) + 3 = 4x + 3$

i.e., $f'(x) = 4x + 3$

11. Compute: $\lim_{x\to 5} \frac{\sqrt{14-x}-3}{x-5} =$

    1. Try Plugging in:
       $\lim_{x\to 5} \frac{\sqrt{14-x}-3}{x-5} = \frac{\sqrt{14-(5)}-3}{(5)-5} = \frac{0}{0}$ No Good - Zero Divide!

    2. Try Factoring and Cancelling:
       $\lim_{x\to 5} \frac{\sqrt{14-x}-3}{x-5} = \lim_{x\to 5} \frac{\sqrt{14-x}-3}{x-5} \cdot \frac{\sqrt{14-x}+3}{\sqrt{14-x}+3} = \lim_{x\to 5} \frac{\left(\sqrt{14-x}\right)^2-(3)^2}{(x-5)\left(\sqrt{14-x}+3\right)}$

       $= \lim_{x\to 5} \frac{(14-x)-9}{(x-5)\left(\sqrt{14-x}+3\right)} = \lim_{x\to 5} \frac{(5-x)}{(x-5)\left(\sqrt{14-x}+3\right)} = \lim_{x\to 5} \frac{-(x-5)}{(x-5)\left(\sqrt{14-x}+3\right)}$

       $= \lim_{x\to 5} \frac{-1}{\left(\sqrt{14-x}+3\right)} = \frac{-1}{\left(\sqrt{14-5}+3\right)} = -\frac{1}{6}$

i.e., $\lim_{x\to 5} \frac{\sqrt{14-x}-3}{x-5} = -\frac{1}{6}$