Integrals and Natural Logarithms #3 - Solutions

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Instructions

Answers appear on the ANSWERS page. Solutions appear on the SOLUTIONS page.

1. Compute: $\int \left(7x^4 + x^3 + 5x + 10\right) dx =$

$\int \left(7x^4 + x^3 + 5x + 10\right) dx = 7\left[\frac{x^5}{5}\right] + \left[\frac{x^4}{4}\right] + 5\left[\frac{x^2}{2}\right] + 10x + C$

i.e., $\int \left(7x^4 + x^3 + 5x + 10\right) dx = \frac{7}{5}x^5 + \frac{1}{4}x^4 + \frac{5}{2}x^2 + 10x + C$
Don't forget the "+C"

2. Compute: $\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx =$

$\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx = 8[\sec(x)] + 5[-\cot(x)] + C$

i.e., $\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx = 8\sec(x) - 5\cot(x) + C$
Don't forget the "+C"

3. Compute: $\int_{x=-1}^{x=1} \left(x^3 + 9x^2 + 3\right) dx =$

$$\int_{x=-1}^{x=1} \underbrace{\left(x^3 + 9x^2 + 3\right)}_{f(x)} dx = \underbrace{\left[\frac{1}{4}x^4 + 3x^3 + 3x\right]_{x=-1}^{x=1}}_{F(x)}$$

$$= \underbrace{\left[\frac{1}{4}(1)^4 + 3(1)^3 + 3(1)\right]}_{F(1)} - \underbrace{\left[\frac{1}{4}(-1)^4 + 3(-1)^3 + 3(-1)\right]}_{F(-1)} = 12$$

i.e., $\int_{x=-1}^{x=1} \left(x^3 + 9x^2 + 3\right) dx = 12$

4. Compute: $\int \sqrt{4x^3+6x}\,\left(6x^2+3\right)dx \underbrace{=}_{\text{Re-write}} \int \left(4x^3+6x\right)^{\frac{1}{2}}\left(6x^2+3\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\left(4x^3+6x\right)^{\frac{1}{2}}$ (A function raised to a power is always a composite function!)

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = \left(4x^3+6x\right)$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(4x^3+6x\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(6x^2+3\right)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = \left(4x^3+6x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+6x \\ \Rightarrow \frac{du}{dx} &= 12x^2+6 \\ \Rightarrow du &= \left(12x^2+6\right)dx \\ \Rightarrow \frac{1}{2}du &= \left(6x^2+3\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(4x^3+6x\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}}\underbrace{\left(6x^2+3\right)dx}_{\frac{1}{2}du} = \int u^{\frac{1}{2}} \cdot \frac{1}{2}du = \frac{1}{2}\int u^{\frac{1}{2}}du$$

4. Integrate (in terms of u).

$$\frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{3}u^{\frac{3}{2}} + C$$

5. Re-express in terms of the original variable, x.

$$\int \sqrt{4x^3+6x}\,\left(6x^2+3\right)dx = \underbrace{\frac{1}{3}\left(4x^3+6x\right)^{\frac{3}{2}} + C}_{\frac{1}{3}u^{\frac{3}{2}}+C}$$

$$\text{i.e., } \int \sqrt{4x^3+6x}\,\left(6x^2+3\right)dx = \frac{1}{3}\left(4x^3+6x\right)^{\frac{3}{2}} + C$$

5. Compute: $\int \sec\left(x^2\right)\tan\left(x^2\right)\, x\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\sec\left(x^2\right)\tan\left(x^2\right)$

   $\nwarrow \quad \nearrow$

   inner — the "outer" is sec ( ) tan ( )

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2 \\ \Rightarrow \frac{du}{dx} &= 2x \\ \Rightarrow du &= 2xdx \\ \Rightarrow \frac{1}{2}du &= xdx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sec\left(x^2\right)\tan\left(x^2\right)}_{\sec(u)\tan(u)} \underbrace{xdx}_{\frac{1}{2}du} = \int \sec(u)\tan(u)\ \frac{1}{2}du = \frac{1}{2}\int \sec(u)\tan(u)\, du$$

4. Integrate (in terms of u).

$$\frac{1}{2}\int \sec(u)\tan(u)\, du = \frac{1}{2}\left[\sec(u)\right] + C = \frac{1}{2}\sec(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sec\left(x^2\right)\tan\left(x^2\right)\ x\, dx = \underbrace{\frac{1}{2}\sec\left(x^2\right) + C}_{\frac{1}{2}\sec(u)+C}$$

i.e., $\int \sec\left(x^2\right)\tan\left(x^2\right)\ x\, dx = \frac{1}{2}\sec\left(x^2\right) + C$

6. Compute: $\int \frac{3x^2+x+2}{2x^3+x^2+4x}\,dx \underbrace{=}_{\text{re-write}} \int \frac{1}{2x^3+x^2+4x}\left(3x^2+x+2\right)dx$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{2x^3+x^2+4x}$ is the same as $\left(2x^3+x^2+4x\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(2x^3+x^2+4x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(2x^3+x^2+4x\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(3x^2+x+2\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(2x^3+x^2+4x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 2x^3+x^2+4x \\ \Rightarrow \frac{du}{dx} &= 6x^2+2x+4 \\ \Rightarrow du &= \left(6x^2+2x+4\right)dx \\ \Rightarrow \frac{1}{2}du &= \left(3x^2+x+2\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{2x^3+x^2+4x}}_{\frac{1}{u}}\underbrace{\left(3x^2+x+2\right)dx}_{\frac{1}{2}du} = \int \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\int\frac{1}{u}du$$

4. Integrate (in terms of u).

$\frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\left[\ln|u|\right] + C = \frac{1}{2}\ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \frac{3x^2+x+2}{2x^3+x^2+4x}\,dx = \underbrace{\frac{1}{2}\ln\left|2x^3+x^2+4x\right| + C}_{\frac{1}{2}\ln|u|+C}$$

i.e., $\int \frac{3x^2+x+2}{2x^3+x^2+4x}\,dx = \frac{1}{2}\ln\left|2x^3+x^2+4x\right| + C$

7. Compute: $\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\tan\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\sec^2\left(x\right)}_{g'(x)} = \frac{\sec^2(x)}{\tan(x)}$$

i.e., $\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right] = \frac{\sec^2(x)}{\tan(x)}$

8. Compute: $\frac{d}{dx}\left[\ln\left(8x^3+5x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(8x^3+5x\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{8x^3+5x}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(24x^2+5\right)}_{g'(x)} = \frac{24x^2+5}{8x^3+5x}$$

i.e., $\frac{d}{dx}\left[\ln\left(8x^3+5x\right)\right] = \frac{24x^2+5}{8x^3+5x}$

9. Compute: $\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] =$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \underbrace{\frac{d}{dx}\left[\ln\left(x\right)+\ln\left(\sin\left(x\right)\right)\right]}_{\ln(ab)=\ln(a)+\ln(b)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \frac{d}{dx}\left[\ln\left(x\right)+\ln\left(\sin\left(x\right)\right)\right] = \left[\frac{1}{x}+\frac{1}{\sin(x)}\cos\left(x\right)\right] = \frac{1}{x} - \cot\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \frac{1}{x} + \cot\left(x\right)$

10. Compute: $\int_{x=0}^{x=1} \left(1-x^2\right)^2 x\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(1-x^2\right)^2$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(1-x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(1-x^2\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(1-x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 1-x^2 \\ \Rightarrow \frac{du}{dx} &= -2x \\ \Rightarrow du &= -2xdx \\ \Rightarrow -\tfrac{1}{2}du &= xdx \end{aligned}$$

   When $x = 0$, $u = 1 - x^2 = 1 - (0)^2 = 1$
   When $x = 1$, $u = 1 - x^2 = 1 - (1)^2 = 0$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{\left(1-x^2\right)^2}_{u^2} \underbrace{xdx}_{-\frac{1}{2}du} = \int_{u=1}^{u=0} u^2 \cdot \left(-\tfrac{1}{2}\right) du = -\tfrac{1}{2} \int_{u=1}^{u=0} u^2 du$$

   Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$-\tfrac{1}{2} \int_{u=1}^{u=0} u^2 du = -\tfrac{1}{2} \left[\tfrac{u^3}{3}\right]_{u=1}^{u=0} = -\tfrac{1}{6} \left[u^3\right]_{u=1}^{u=0} = \underbrace{-\frac{1}{6}(0)^3}_{F(0)} - \underbrace{\left(-\frac{1}{6}(1)^3\right)}_{F(1)} = 0 - \left(-\tfrac{1}{6}\right) = \tfrac{1}{6}$$

i.e., $\int_{x=0}^{x=1} \left(1-x^2\right)^2 x\,dx = \frac{1}{6}$