MTH 1125 (12 pm) Test #3 - Solutions

Fall 2024

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 - 3x^2 - 36x + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 6x^2 - 6x - 36$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 6x^2 - 6x - 36 = 0$

   $\Rightarrow x^2 - x - 6 = 0$

   $\Rightarrow (x + 2)(x - 3) = 0$

   $\Rightarrow x = -2$ and $x = 3$ are critical numbers.

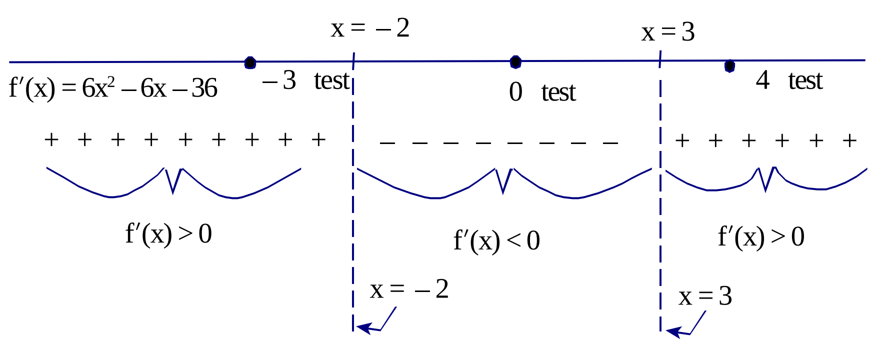
   b. "Type b" $(f'(c)$ is undefined$)$

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

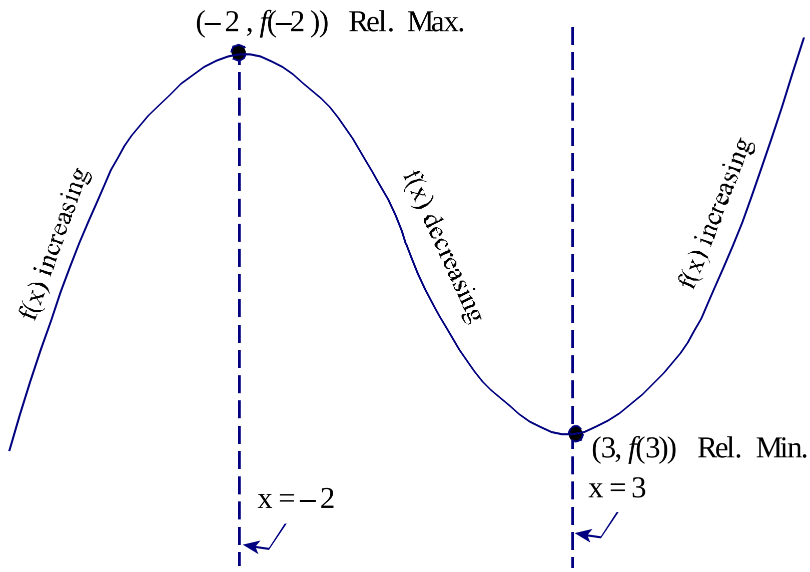
3. Pick a "test point" from each interval to plug into $f'(x)$



$f(x)$ is **increasing** on the interval(s) $(-\infty, -2)$ and $(3, \infty)$
(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-2, 3)$
(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-2, f(-2)) = (-2, 46)$

Rel Min $(3, f(3)) = (3, -79)$

2. $f(x) = x^4 - 4x^3 - 48x^2 + 4x + 2$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = 4x^3 - 12x^2 - 96x + 4$

      $f''(x) = 12x^2 - 24x - 96$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 - 24x - 96 = 0$

         $\Rightarrow x^2 - 2x - 8 = 0$

         $\Rightarrow (x+2)(x-4) = 0$

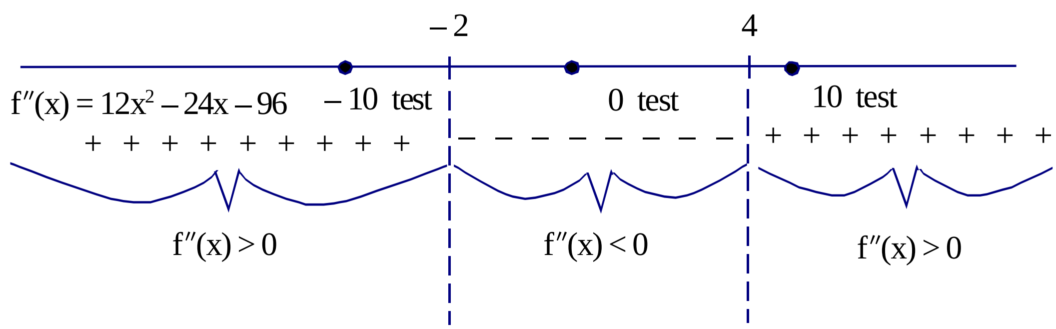
         $x = -2, 4$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined$)$

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -2)$ and $(4, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-2, 4)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -2$ and $x = 4$, the points:
$(-2, f(-2)) = (-2, -150)$
and
$(4, f(4)) = (4, -750)$ **are** points of inflection.

3. $f(x) = x^3 - 3x^2 - 9x + 2$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 3x^2 - 6x - 9$

a. "Type a" $(f'(x) = 0)$

$f'(x) = 3x^2 - 6x - 9 = 0$

$\Rightarrow 3(x^2 - 2x - 3) = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = -1, 3$ are "type a" critical numbers

Since $x = 3$ is not in the interval $[-2, 2]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = 0$

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = 7 \leftarrow$ Abs Max Value

$f(2) = (2)^3 - 3(2)^2 - 9(2) + 2 = -20 \leftarrow$ Abs Min Value

The Abs Max Value is 7 (attained at $x = -1$) The Abs Min Value is -20 (attained at $x = 2$)

4. $f(x) = 3x^{\frac{12}{5}} - 18x^{\frac{2}{5}} + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

   $f'(x) = \frac{36}{5}x^{\frac{7}{5}} - \frac{36}{5}x^{-\frac{3}{5}} = \frac{36x^{\frac{7}{5}}}{5} - \frac{36}{5x^{\frac{3}{5}}} = \frac{36x^{\frac{7}{5}}}{5}\frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} - \frac{36}{5x^{\frac{3}{5}}} = \frac{36x^2-36}{5x^{\frac{3}{5}}}$

   i.e., $f'(x) = \frac{36x^2-36}{5x^{\frac{3}{5}}}$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = \frac{36x^2-36}{5x^{\frac{3}{5}}} = 0$

   $\Rightarrow 36x^2 - 36 = 0$

   $\Rightarrow x^2 - 1 = 0$

   $\Rightarrow (x+1)(x-1) = 0$

   $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

   b. "Type b" $(f'(c)$ is undefined$)$

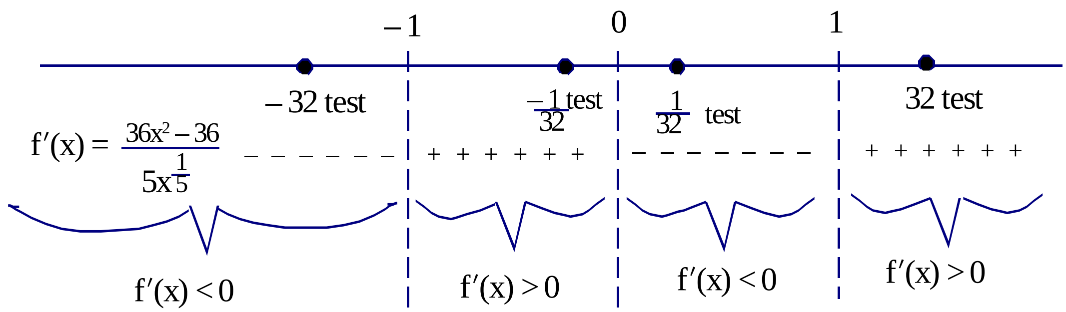
   Look for x-value that causes division by zero.

   $\Rightarrow 5x^{\frac{3}{5}} = 0$

   $\Rightarrow x = 0$ "type b" critical number

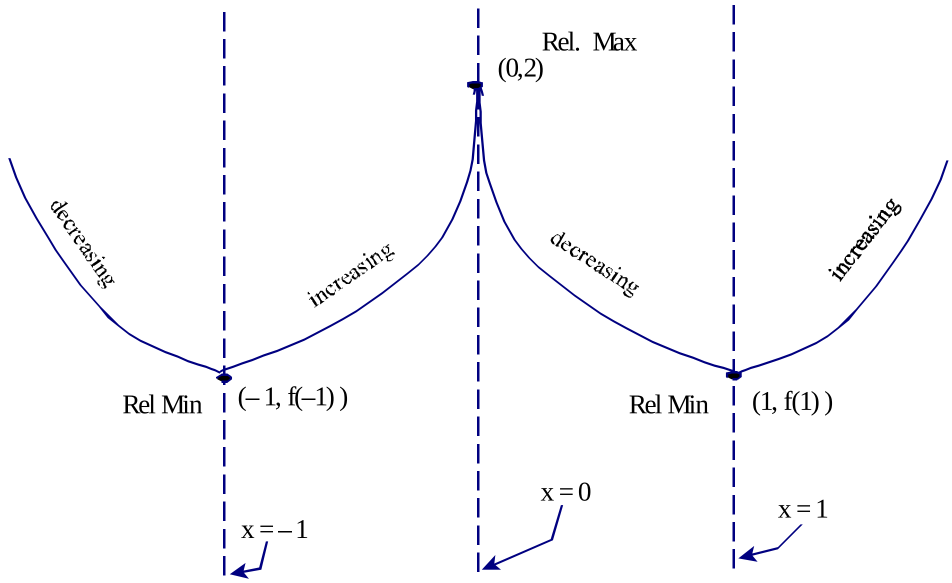
2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



> $f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$
>
> (because $f'(x)$ is positive on these intervals)
>
> $f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$
>
> (because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

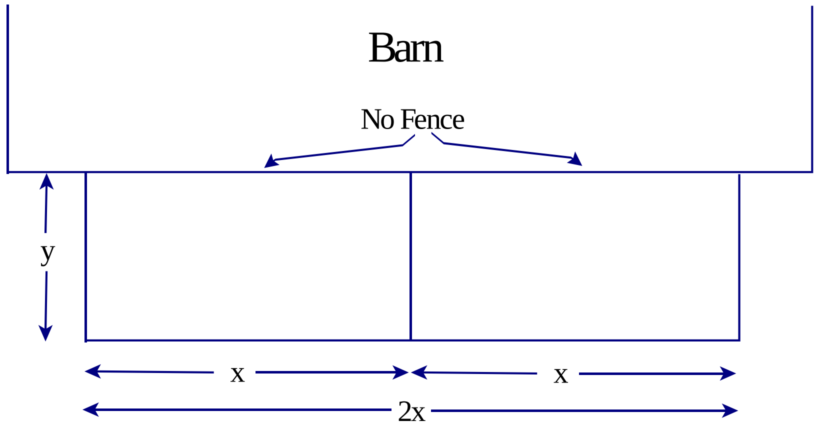


Rel Minimums: $(-1, f(-1)) = \left(-1, 3(-1)^{\frac{12}{5}} - 18(-1)^{\frac{2}{5}} + 2\right) = (-1, -13)$

and $(1, f(1)) = \left(1, 3(1)^{\frac{12}{5}} - 18(1)^{\frac{2}{5}} + 2\right) = (-1, -13)$

Rel Maximum: $(0, f(0)) = (0, 2)$

5. Farmer Joe has 150 feet of wire fencing. He will use the fencing to make a rectangular pen. His barn will form one side of the pen, so no wire fencing will be used on that side. In addition, he will use some of the fencing to partition the pen into two smaller pens of similar shape and equal area. (See below) What should the overall dimensions of the pen be, in order for the enclosed area to be as large as possible?



Solution: Version 1 (Express area A as a function of x)

i. Declare what it is that you want to maximize/minimize — give it a name.

   Maximize enclosed area, $A = 2xy$

   Draw a picture

   (Done)

ii. Express A as a function of one variable. (Refer to a restriction stated in the problem.)

   Restriction: Use 150 ft of fencing

   $\Rightarrow 2x + 3y = 150$ ft

   $\Rightarrow 3y = 150$ ft $- 2x$

   $\Rightarrow y = 50$ ft $- \frac{2}{3}x$

   Plug this into the equation $A = 2xy$

   $\Rightarrow A = 2x\left(50 \text{ ft } - \frac{2}{3}x\right) = 100 \text{ ft } x - \frac{4}{3}x^2$

   i.e., $A(x) = 100 \text{ ft } x - \frac{4}{3}x^2$

iii. Determine the restrictions on the independent variable x.

   $0 \text{ ft} \leq x \leq \frac{150}{2}$ ft

   i.e., $0 \text{ ft} \leq x \leq 75$ ft

iv. Maximize/minimize the function, using the techniques of calculus

Since $A(x)$ is continuous on the closed, finite interval $[0 \text{ ft}, 75 \text{ ft}]$, we can use the Absolute Max/Min Value Test.

Compute $A'(x)$ and find the critical numbers

$A'(x) = 100 \text{ ft} - \frac{8}{3}x$

Type a: set $A'(x) = 100 \text{ ft} - \frac{8}{3}x = 0$

$\Rightarrow \frac{8}{3}x = 100 \text{ ft}$

$\Rightarrow x = \frac{300}{8} \text{ ft} = \frac{75}{2} \text{ ft}$ (critical number)

Type b: No Type b critical numbers

Plug endpoints and critical numbers into $A(x)$, the original function.

$A(0 \text{ ft}) = 100 \text{ ft } (0 \text{ ft}) - \frac{4}{3}(0 \text{ ft})^2 = 0 \text{ ft}^2$

$A\left(\frac{75}{2} \text{ ft}\right) = 100 \text{ ft } \left(\frac{75}{2} \text{ ft}\right) - \frac{4}{3}\left(\frac{75}{2} \text{ ft}\right)^2 = 1875 \text{ ft}^2 \leftarrow$ Abs Max Area

$A(75 \text{ ft}) = 100 \text{ ft } (75 \text{ ft}) - \frac{4}{3}(75 \text{ ft})^2 = 0 \text{ ft}^2$

v. Make sure we've answered the original question:

"What dimensions should be used . . . "

Length = 2x = $2\left(\frac{75}{2} \text{ ft}\right) = 75 \text{ ft}$

Width = $y = 50 \text{ ft} - \frac{2}{3}x = 50 \text{ ft} - \frac{2}{3}\left(\frac{75}{2} \text{ ft}\right) = 25 \text{ ft}$

Length = 2x = 75 ft
Width = $y = 25$ ft

Solution: Version 2 (Express area A as a function of y)

(Appears on the next page)

Solution: Version 2 (Express area A as a function of y)

i. Declare what it is that you want to maximize/minimize — give it a name.

   Maximize enclosed area, $A = 2xy$

   Draw a picture

   (Done)

ii. Express A as a function of one variable. (Refer to a restriction stated in the problem.)

   Restriction: Use 150 ft of fencing

   $\Rightarrow 2x + 3y = 150$ ft

   $\Rightarrow 2x = 150$ ft $- 3y$

   Plug this into the equation $A = 2xy$

   $\Rightarrow A = (150 \text{ ft } - 3y)\, y = 150 \text{ ft } y - 3y^2$

   i.e., $A(x) = 150 \text{ ft } y - 3y^2$

iii. Determine the restrictions on the independent variable y.

   $0 \text{ ft} \leq y \leq 50 \text{ ft}$

iv. Maximize/minimize the function, using the techniques of calculus

   Since $A(y)$ is continuous on the closed, finite interval $[0 \text{ ft}, 50 \text{ ft}]$, we can use the Absolute Max/Min Value Test.

   Compute $A'(y)$ and find the critical numbers

   $A'(y) = 150 \text{ ft } -6y$

   Type a: set $A'(y) = 150 \text{ ft } -6y = 0$

   $\Rightarrow 6y = 150$ ft

   $\Rightarrow y = 25$ ft (critical number)

   Type b: No Type b critical numbers

Plug endpoints and critical numbers into $A(x)$, the original function.

$A(0\text{ ft}) = 150\text{ ft }(0\text{ ft}) - 3(0\text{ ft})^2 = 0\text{ ft}^2$

$A(25\text{ ft}) = 150\text{ ft }(25\text{ ft}) - 3(25\text{ ft})^2 = 1875\text{ ft}^2 \leftarrow$ Abs Max Area

$A(50\text{ ft}) = 150\text{ ft }(50\text{ ft}) - 3(50\text{ ft})^2 = 0\text{ ft}^2$

v. Make sure we've answered the original question:

"What dimensions should be used . . . "

Width $= y = 25$ ft

Length $= 2\text{x} = 150\text{ ft} - 3y = 150\text{ ft} - 3(25\text{ ft}) = 75$ ft

Length $= 2\text{x} = 75$ ft
Width $= y = 25$ ft

EXTRA! (Wow! 10 points!)

In the exercise below, [1]Deterimine the intervals on which $f(x)$ is increasing/decreasing
[2]Identify all relative maximums and minimums
[3]Deterimine the intervals on which $f(x)$ is CCU/CCD
[4]Identify all points of inflections
[5]Graph $f(x)$

$f(x) = x^3 + 3x^2 - 9x + 5$

(Increasing/Decreasing - Max/Mins)

1. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 + 6x - 9$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 3x^2 + 6x - 9 = 0$

      $\Rightarrow 3x^2 + 6x - 9 = 0$

      $\Rightarrow x^2 + 2x - 3 = 0$

      $\Rightarrow (x+3)(x-1) = 0$

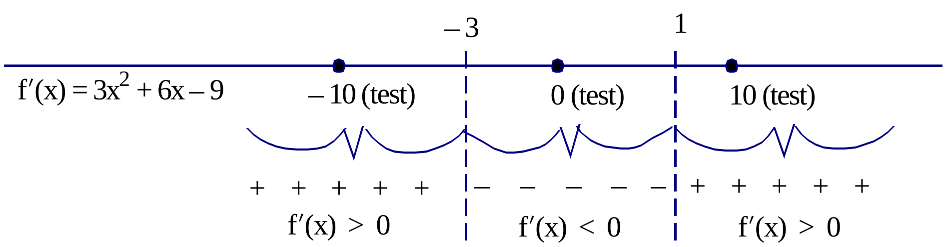
      $\Rightarrow x = -3;\ x = 1$ critical numbers

   b. "Type b" $(f'(c)$ undefined)

      There are none.

2. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

3. From each interval select a "test point" to plug into $f'(x)$



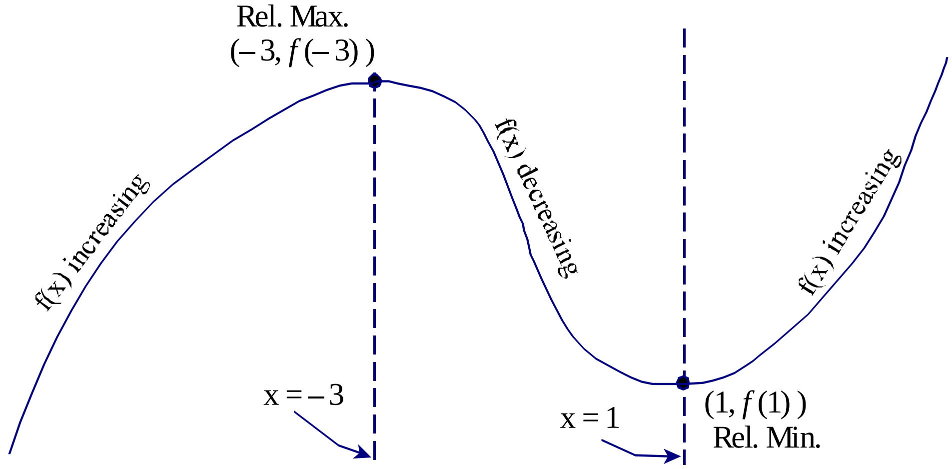
$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(1, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-3, 1)$

(Because $f'(x)$ is negative on this interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-3, f(-3)) = (-3, 32)$
Rel Min $(1, f(1)) = (1, 0)$

(Concave Up/Concave Down - Points of inflection)

i. Compute $f''(x)$ and find possible points of inflection

$f'(x) = 3x^2 + 6x - 9$

$f''(x) = 6x + 6$

a. "Type a" $(f''(c) = 0)$

Set $f''(x) = 6x + 6 = 0$

$\Rightarrow 6x + 6 = 0$

$\Rightarrow x + 1 = 0$

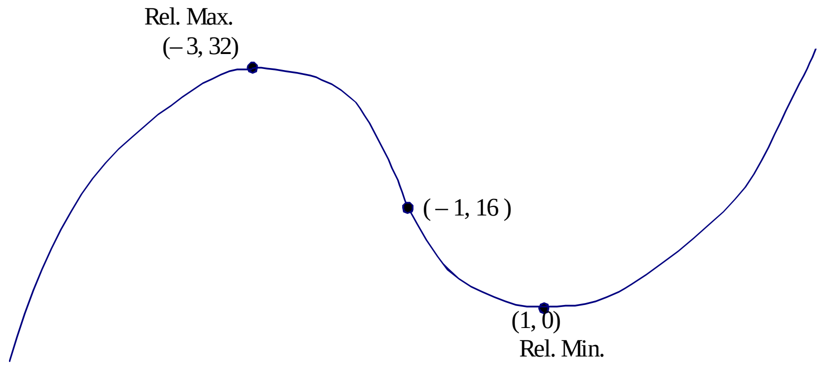
$\Rightarrow x = -1$ possible point of inflection

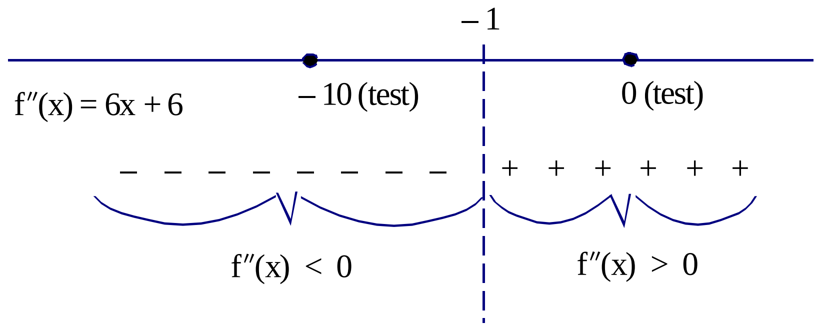
b. "Type b" ($f''(c)$ undefined)

There are none.

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

iii. From each interval select a "test point" to plug into $f''(x)$





$f(x)$ is **concave down** on the interval $(-\infty, -1)$

(Because $f''(x) < 0$ on this interval)

$f(x)$ is **concave up** on the interval $(-1, \infty)$

(Because $f''(x) > 0$ on this interval)

Since $f(x)$ changes concavity at $x = -1$, the point:

$(-1, f(-1)) = (-1, 16)$ *is* a point of inflection

Graph of $f(x) = 2x^3 - 12x^2 + 18x - 3$