Integrals and Natural Logarithms #2 - Solutions

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Instructions

Answers appear on the ANSWERS page. Solutions appear on the SOLUTIONS page.

1. Compute: $\int \left(2x^4 + 6x^3 + 3x + 6\sqrt{x} + 2\right) dx =$

   (Re-write) $\int \left(2x^4 + 6x^3 + 3x + 6x^{\frac{1}{2}} + 2\right) dx = 2\left[\frac{x^5}{5}\right] + 6\left[\frac{x^4}{4}\right] + 3\left[\frac{x^2}{2}\right] + 6\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] + 2x + C$

   $= \frac{2}{5}x^5 + \frac{3}{2}x^4 + \frac{3}{2}x^2 + 4x^{\frac{3}{2}} + 2x + C$

   i.e., $\int \left(2x^4 + 6x^3 + 3x + 6\sqrt{x} + 2\right) dx = \frac{2}{5}x^5 + \frac{3}{2}x^4 + \frac{3}{2}x^2 + 4x^{\frac{3}{2}} + 2x + C$
   Don't forget the "+C"

2. Compute: $\int \left(2\cos(x) + 5\sec^2(x)\right) dx =$

   $\int \left(2\cos(x) + 5\sec^2(x)\right) dx = 2[\sin(x)] + 5[\tan(x)] + C$

   i.e., $\int \left(2\cos(x) + 5\sec^2(x)\right) dx = 2[\sin(x)] + 5[\tan(x)] + C$
   Don't forget the "+C"

3. Compute: $\int_{x=0}^{x=2} \left(2x^3 + 3x^2 + 2\right) dx =$

   $$\int_{x=0}^{x=2} \underbrace{\left(2x^3 + 3x^2 + 2\right)}_{f(x)} dx = \underbrace{\left[\frac{1}{2}x^4 + x^3 + 2x\right]_{x=0}^{x=2}}_{F(x)}$$

   $$= \underbrace{\left[\frac{1}{2}(2)^4 + (2)^3 + 2(2)\right]}_{F(2)} - \underbrace{\left[\frac{1}{2}(0)^4 + (0)^3 + 2(0)\right]}_{F(0)} = 20$$

   i.e., $\int_{x=0}^{x=2} \left(2x^3 + 3x^2 + 2\right) dx = 20$

4. Compute: $\int \left(2x^3+2x\right)^5 \left(3x^2+1\right) dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(2x^3+2x\right)^5$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(2x^3+2x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(2x^3+2x\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(3x^2+1\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(2x^3+2x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 2x^3+2x \\ \Rightarrow \frac{du}{dx} &= 6x^2+2 \\ \Rightarrow du &= \left(6x^2+2\right) dx \\ \Rightarrow \tfrac{1}{2} du &= \left(3x^2+1\right) dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(2x^3+2x\right)^5}_{u^5} \underbrace{\left(3x^2+1\right) dx}_{\frac{1}{2}du} = \int u^5 \tfrac{1}{2} du = \tfrac{1}{2} \int u^5 du$$

4. Integrate (in terms of u).

$$\tfrac{1}{2} \int u^5 du = \tfrac{1}{2}\left[\tfrac{u^6}{6}\right] + C = \tfrac{1}{12}u^6 + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(2x^3+2x\right)^5 \left(3x^2+1\right) dx = \underbrace{\frac{1}{12}\left(2x^3+2x\right)^6 + C}_{\frac{1}{12}u^6 + C}$$

i.e., $\int \left(2x^3+2x\right)^5 \left(3x^2+1\right) dx = \frac{1}{12}\left(2x^3+2x\right)^6 + C$

5. Compute: $\int \cos\left(3x^2\right)\, x\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\cos\left(3x^2\right)$

   $\nearrow \qquad \nwarrow$

   outer inner

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 3x^2$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{3x^2}_{\text{function}} \; - - - - \rightarrow \; \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 3x^2$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2 \\ \Rightarrow \frac{du}{dx} &= 6x \\ \Rightarrow du &= 6x\, dx \\ \Rightarrow \frac{1}{6} du &= x\, dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\cos\left(3x^2\right)}_{\cos(u)} \underbrace{x\, dx}_{\frac{1}{6}du} = \int \cos(u)\ \frac{1}{6} du = \frac{1}{6} \int \cos(u)\, du$$

4. Integrate (in terms of u).

$$\frac{1}{6} \int \cos(u)\, du = \frac{1}{6} [\sin(u)] + C = \frac{1}{6} \sin(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \cos\left(3x^2\right)\, x\, dx = \underbrace{\frac{1}{6} \sin\left(3x^2\right) + C}_{\frac{1}{6}\sin(u)+C}$$

i.e., $\int \cos\left(3x^2\right)\, x\, dx = \frac{1}{6} \sin\left(3x^2\right) + C$

6. Compute: $\int \frac{x}{3x^2+6} dx =$

$\int \frac{x}{3x^2+6} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3x^2+6}\, x\, dx$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{3x^2+6}$ is the same as $\left(3x^2+6\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^2+6\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^2+6\right)}_{\text{function}} - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^2+6\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2+6 \\ \Rightarrow \frac{du}{dx} &= 6x \\ \Rightarrow du &= 6x dx \\ \Rightarrow \frac{1}{6} du &= x dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{3x^2+6}}_{\frac{1}{u}} \underbrace{x\,dx}_{\frac{1}{6}du} = \int \frac{1}{u} \cdot \frac{1}{6} du = \frac{1}{6} \int \frac{1}{u} du$$

4. Integrate (in terms of u).

   $\frac{1}{6} \int \frac{1}{u} du = \frac{1}{6} \left[\ln|u|\right] + C = \frac{1}{6} \ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \frac{x}{3x^2+6} dx = \underbrace{\frac{1}{6} \ln\left|3x^2+6\right| + C}_{\frac{1}{6}\ln|u|+C}$$

$$\text{i.e., } \int \frac{x}{3x^2+6} dx = \frac{1}{6} \ln\left|3x^2+6\right| + C$$

7. Compute: $\frac{d}{dx}\left[\ln\left(\cos\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\cos\left(x\right)\right)\right]}_{\frac{d}{dx}\left[\ln(g(x))\right]} = \underbrace{\frac{1}{\cos\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(-\sin\left(x\right)\right)}_{g'(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan\left(x\right)$$

> i.e., $\frac{d}{dx}\left[\ln\left(\cos\left(x\right)\right)\right] = -\frac{\sin(x)}{\cos(x)} = -\tan\left(x\right)$

8. Compute: $\frac{d}{dx}\left[\ln\left(5x^2+5x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(5x^2+5x\right)\right]}_{\frac{d}{dx}\left[\ln(g(x))\right]} = \underbrace{\frac{1}{5x^2+5x}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(10x+5\right)}_{g'(x)} = \frac{10x+5}{5x^2+5x}$$

> i.e., $\frac{d}{dx}\left[\ln\left(5x^2+5x\right)\right] = \frac{10x+5}{5x^2+5x}$

9. Compute: $\frac{d}{dx}\left[\ln\left(x\sqrt{x^2-1}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left(x\left(x^2-1\right)^{\frac{1}{2}}\right)\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(x\left(x^2-1\right)^{\frac{1}{2}}\right)\right] = \underbrace{\frac{d}{dx}\left[\ln\left(x\right) + \ln\left[\left(x^2-1\right)^{\frac{1}{2}}\right]\right]}_{\ln(ab)=\ln(a)+\ln(b)} = \underbrace{\frac{d}{dx}\left[\ln\left(x\right) + \frac{1}{2}\ln\left(x^2-1\right)\right]}_{\ln(a^n)=n\ln(a)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(x\sqrt{x^2-1}\right)\right] = \frac{d}{dx}\left[\ln\left(x\right) + \frac{1}{2}\ln\left(x^2-1\right)\right] = \frac{d}{dx}\left[\ln\left(x\right)\right] + \frac{d}{dx}\left[\frac{1}{2}\ln\left(x^2-1\right)\right]$$

$$= \frac{1}{x} + \frac{1}{2}\frac{1}{x^2-1}\cdot 2x = \frac{1}{x} + \frac{x}{x^2-1}$$

> i.e., $\frac{d}{dx}\left[\ln\left(x\sqrt{x^2-1}\right)\right] = \frac{1}{x} + \frac{x}{x^2-1}$

10. Compute: $\int_{x=0}^{x=1} \left(x^2+1\right)^3 x\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^2+1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^2+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2+1\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^2+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2+1 \\ \Rightarrow \frac{du}{dx} &= 2x \\ \Rightarrow du &= 2xdx \\ \Rightarrow \frac{1}{2}du &= xdx \end{aligned}$$

When $x=0$, $u = x^2+1 = (0)^2+1 = 1$
When $x=1$, $u = x^2+1 = (1)^2+1 = 2$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{\left(x^2+1\right)^3}_{u^3} \underbrace{x\,dx}_{\frac{1}{2}du} = \int_{u=1}^{u=2} u^3 \cdot \frac{1}{2}du = \frac{1}{2}\int_{u=1}^{u=2} u^3 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{2}\int_{u=1}^{u=2} u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_{u=1}^{u=2} = \frac{1}{8}\left[u^4\right]_{u=1}^{u=2} = \frac{1}{8}\left(\underbrace{(2)^4}_{F(2)} - \underbrace{(1)^4}_{F(1)}\right) = \frac{15}{8}$$

i.e., $\int_{x=0}^{x=1} \left(x^2+1\right)^3 x\,dx = \frac{15}{8}$