MTH 3311 Practice Test #2 - Solutions

SUMMER 2016

Pat Rossi Name ____________________

1. (p. 109 #2) Water at temperature 10° C takes 5 minutes to warm up to 20° C in a room at temperature 40° C

   (a) Find the temperature after 20 minutes; after 30 minutes

   Let T be the temperature of the water at time $t \geq 0$. Newtons law of heating/cooling tells us that the rate ($\frac{dT}{dt}$) at which the water heats up or cools down is proportional to the difference between the temperature of the water and the temperature of the surrounding environment (room temperature), T_r.

   i.e., $\frac{dT}{dt} = k(T_r - T)$, where k is the constant of proportionality.

   Separating the variables, we have:

   $\frac{1}{(T_r - T)} dT = kdt$

   $\int \frac{1}{(T_r - T)} dT = \int kdt$

   $\Rightarrow \ln|T_r - T| = kt + C$

   $\Rightarrow \ln(T_r - T) = kt + C$ (no absolute value bars needed, since $T_r - T > 0$.)

   $\Rightarrow e^{\ln(T_r - T)} = e^{kt+C}$

   $\Rightarrow T_r - T = Ce^{kt}$

   $\Rightarrow T = T_r - Ce^{kt}$

   $\Rightarrow T = 40° - Ce^{kt}$ (Room temperature is 40°)

   Recall: At time $t = 0$ min, $T = 10°$

   $\Rightarrow 10° = 40° - Ce^{k(0 \text{ sec})}$

   $\Rightarrow 10° = 40° - C$

   $\Rightarrow C = 40° - 10° = 30°$

   Hence, $T = 40° - 30°e^{kt}$

   Recall Also: At time $t = 5$ min, $T = 20°$

   $\Rightarrow 20° = 40° - 30°e^{k(5 \text{ min})}$

   $\Rightarrow 30°e^{k(5 \text{ min})} = 40° - 20°$

   $\Rightarrow 30°e^{k(5 \text{ min})} = 20°$

   $\Rightarrow e^{k(5 \text{ min})} = \frac{2}{3}$

   $\Rightarrow \ln\left(e^{k(5 \text{ min})}\right) = \ln\left(\frac{2}{3}\right)$

   $\Rightarrow k(5 \text{ min}) = \ln\left(\frac{2}{3}\right)$

   $\Rightarrow k = \frac{-0.081093}{\text{min}}$

   $\Rightarrow T = 40° - 30°e^{\frac{-0.081093}{\text{min}}t}$

The temperature after 20 minutes is given by:

$$T = 40^\circ - 30^\circ e^{\frac{-0.081093}{\text{min}}(20\ \text{min})} = 34.074^\circ$$

$$\boxed{T\,(20\ \text{min}) = 34.074^\circ}$$

The temperature after 30 minutes is given by:

$$T = 40^\circ - 30^\circ e^{\frac{-0.081093}{\text{min}}(30\ \text{min})} = 37.366^\circ$$

$$\boxed{T\,(30\ \text{min}) = 37.366^\circ}$$

(b) When will the temperature be 25° C?

$$\Rightarrow 25^\circ = 40^\circ - 30^\circ e^{\frac{-0.081093}{\text{min}}t}$$

$$\Rightarrow -15^\circ = -30^\circ e^{\frac{-0.081093}{\text{min}}t}$$

$$\Rightarrow \frac{1}{2} = e^{\frac{-0.081093}{\text{min}}t}$$

$$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{\frac{-0.081093}{\text{min}}t}\right)$$

$$\Rightarrow \ln\left(\frac{1}{2}\right) = \frac{-0.081093}{\text{min}}t$$

$$\Rightarrow t = \frac{\ln\left(\frac{1}{2}\right)}{\left(\frac{-0.081093}{\text{min}}\right)} = 8.548\ \text{min}$$

$$\boxed{\text{At } t = 8.548\ \text{min},\ T = 25^\circ\ \text{C}}$$

2. (p. 109 #5) If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% to disappear?

   Recall: The rate $\left(\frac{dA}{dt}\right)$ at which a radioactive substance decays is proportional to the amount A of the substance present.

   i.e., $\frac{dA}{dt} = kA$

   Separating the variables, we have:

   $\frac{1}{A}dA = kdt$

   $\Rightarrow \int \frac{1}{A}dA = \int kdt$

   $\Rightarrow \ln|A| = kt + C$

   $\Rightarrow \ln(A) = kt + C$ (No absolute value bars needed since $A > 0$)

   $\Rightarrow e^{\ln(A)} = e^{kt+C}$

   $\Rightarrow A = Ce^{kt}$

   Let A_0 be the amount of the substance present at time $t = 0$ days.

   Then $A_0 = A(0 \text{ days}) = Ce^{k(0 \text{ days})}$

   $\Rightarrow A_0 = C$

   $\Rightarrow A = A_0 e^{kt}$

   Recall Also: At time $t = 10$ days, $A = 0.7A_0$

   i.e., $0.7A_0 = A(10 \text{ days}) = A_0 e^{k(10 \text{ days})}$

   $\Rightarrow 0.7A_0 = A_0 e^{k(10 \text{ days})}$

   $\Rightarrow 0.7 = e^{k(10 \text{ days})}$

   $\Rightarrow \ln(0.7) = \ln\left(e^{k(10 \text{ days})}\right)$

   $\Rightarrow \ln(0.7) = k(10 \text{ days})$

   $\Rightarrow k = \frac{\ln(0.7)}{10 \text{ days}} = \frac{-0.03567}{\text{days}}$

   $\Rightarrow A = A_0 e^{\frac{-0.03567}{\text{days}}t}$

   We want: t when $A = 0.1A_0$

   $\Rightarrow A = 0.1A_0 = A_0 e^{\frac{-0.03567}{\text{days}}t}$

   i.e., $0.1A_0 = A_0 e^{\frac{-0.03567}{\text{days}}t}$

   $\Rightarrow 0.1 = e^{\frac{-0.03567}{\text{days}}t}$

   $\Rightarrow \ln(0.1) = \ln\left(e^{\frac{-0.03567}{\text{days}}t}\right)$

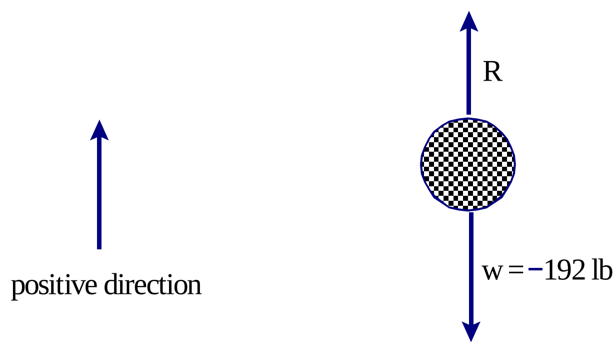
   $\Rightarrow \ln(0.1) = \frac{-0.03567}{\text{days}}t$

$\Rightarrow t = \frac{\ln(0.1)}{\left(\frac{-0.03567}{\text{days}}\right)} = 64.552 \text{ days}$

i.e., When $A = 0.1A_0$, $t = 64.552$ days

3. (p. 79 #7) A 192 lb weight has limiting velocity 16 ft/sec., when falling in air, which provides a resisting force proportional to the weight's instantaneous velocity. If the weight starts from rest:

We draw a force diagram of the object:



The net force on the object is $R + w$

Recall: The air provides a resisting force R proportional to the weight's instantaneous velocity.

i.e., $R = kv$

Also, the weight (which is a force) is given by $w = mg = -192$ lb., where $g = -\frac{32 \text{ ft}}{\text{sec}^2}$ is the acceleration due to earth's gravity.

Recall: the force F on the object is given by
$F = ma$ where a is the object's acceleration.

Since $a = \frac{dv}{dt}$, we have:

$F = ma = \frac{w}{g}\frac{dv}{dt}$

i.e., $F = \frac{w}{g}\frac{dv}{dt}$

From our force diagram, $F = R + w = kv + w$

i.e., $F = kv + w$

This gives us: $\frac{w}{g}\frac{dv}{dt} = kv + w$

Separating the variables, we have:

$\frac{1}{kv+w} dv = \frac{g}{w} dt$

$\Rightarrow \frac{1}{k} \ln|kv + w| = \frac{g}{w} t + C$

$\ln|kv + w| = \frac{kg}{w} t + C$

$\Rightarrow e^{\ln|kv+w|} = e^{\frac{kg}{w}t + C}$

$\Rightarrow |kv + w| = Ce^{\frac{kg}{w}t}$

$\Rightarrow -(kv + w) = Ce^{\frac{kg}{w}t}$

$\Rightarrow -kv - w = Ce^{\frac{kg}{w}t}$

$\Rightarrow -kv = w + Ce^{\frac{kg}{w}t}$

$\Rightarrow v = -\frac{w}{k} + Ce^{\frac{kg}{w}t}$

Recall: "The weight starts from rest." (i.e. at $t = 0$ sec, $v = 0$ ft/sec)

$0\ \frac{\text{ft}}{\text{sec}} = -\frac{w}{k} + Ce^{\frac{kg}{w}(0\text{ sec})} = -\frac{w}{k} + C$

i.e., $0\ \frac{\text{ft}}{\text{sec}} = -\frac{w}{k} + C$

$\Rightarrow \frac{w}{k} = C$

$\Rightarrow v = -\frac{w}{k} + \frac{w}{k}e^{\frac{kg}{w}t}$

Recall: The weight has limiting velocity 16 ft/sec. (in the downward (or negative) direction.

$\Rightarrow \lim_{t\to\infty} v = \lim_{t\to\infty}\left(-\frac{w}{k} + \frac{w}{k}e^{\frac{kg}{w}t}\right) = -16\ \frac{\text{ft}}{\text{sec}}$

But how do we compute $\lim_{t\to\infty} e^{\frac{kg}{w}t}$, without knowing what $\frac{kg}{w}$ is?

Note that given our orientation of "up" being the positive direction, the constants g and w are negative.

Note also that $R = kv$ is a force in the positive ("upward") direction. Since the velocity v is always negative (the object goes *down,* not *up*), the constant k must also be negative in order for R to be positive.

The Point: The constants $g, w,$ and k are all negative. Hence, the constant $\frac{kg}{w}$ must be negative as well. This is all we need to know for now.

Hence, $\lim_{t\to\infty} e^{\frac{kg}{w}t} = 0$ (because the coefficient of t is negative.)

Thus, we have: $\lim_{t\to\infty} v = \lim_{t\to\infty}\left(-\frac{w}{k} + \frac{w}{k}e^{\frac{kg}{w}t}\right) = -16\ \frac{\text{ft}}{\text{sec}}$

$\Rightarrow -\frac{w}{k} = -16\ \frac{\text{ft}}{\text{sec}}$

We learn two things from this equation:

First, $\frac{w}{k} = 16\ \frac{\text{ft}}{\text{sec}}$

Hence, $v = -\frac{w}{k} + \frac{w}{k}e^{\frac{kg}{w}t} = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{\frac{kg}{w}t}$

i.e., $v = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{\frac{kg}{w}t}$

Second, $\frac{w}{k} = 16\ \frac{\text{ft}}{\text{sec}}$

$\Rightarrow k = \frac{w}{16\frac{\text{ft}}{\text{sec}}} = \frac{-192\text{ lb}}{16\frac{\text{ft}}{\text{sec}}} = -12\frac{\text{lb sec}}{\text{ft}}$

i.e., $k = -12\frac{\text{lb sec}}{\text{ft}}$

Thus, $v = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{\frac{kg}{w}t} = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{-\frac{2}{\text{sec}}t}$

i.e., $v = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{-\frac{2}{\text{sec}}t}$

(a) Find the velocity of the weight after 1 second.

$$v\,(1\text{ sec}) = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{-\frac{2}{\text{sec}}(1\text{ sec})} = -13.835\ \frac{\text{ft}}{\text{sec}}$$

$$\boxed{v\,(1\text{ sec}) = -13.835\ \frac{\text{ft}}{\text{sec}}}$$

(b) How long is it before the velocity of the weight is 15 ft/sec?

We want t when $v\,(t) = -15\ \frac{\text{ft}}{\text{sec}}$

$$\Rightarrow -15\ \frac{\text{ft}}{\text{sec}} = -16\frac{\text{ft}}{\text{sec}} + 16\frac{\text{ft}}{\text{sec}}e^{-\frac{2}{\text{sec}}t}$$

$$\Rightarrow -15 = -16 + 16e^{-\frac{2}{\text{sec}}t}$$

$$\Rightarrow 1 = 16e^{-\frac{2}{\text{sec}}t}$$

$$\Rightarrow \frac{1}{16} = e^{-\frac{2}{\text{sec}}t}$$

$$\Rightarrow \ln\left(\frac{1}{16}\right) = \ln\left(e^{-\frac{2}{\text{sec}}t}\right)$$

$$\Rightarrow \ln\left(\frac{1}{16}\right) = -\frac{2}{\text{sec}}t$$

$$\Rightarrow t = -\frac{1}{2}\ln\left(\frac{1}{16}\right)\text{ sec} = 1.386\text{ sec}$$

$$\boxed{\text{When } v = -15\ \frac{\text{ft}}{\text{sec}},\ t = 1.386\text{ sec}}$$