MTH 6610 - History of Math Reading Assignment #4 - Answers

Term V - 2024

Pat Rossi   Name ____________________

Instructions. Read pages 105 - 137 to find the answers to these questions in your reading.

1. What formula for generating Pythagorean Triples generates sides of right triangles having the characteristic that the hypotenuse is of length one greater than one of the legs?

   $x = 2n + 1$; $y = 2n^2 + 2n$; $z = 2n^2 + 2n + 1$

2. What formula for generating Pythagorean Triples generates sides of right triangles having the characteristic that the hypotenuse is of length two greater than one of the legs?

   $x = 2n$; $y = n^2 - 1$; $z = n^2 + 1$

3. In Euclid's book X of *Elements,* there is a formula for generating Pythagorean triples. What is the formula, and what is so special about it?

   $x = 2mn$; $y = m^2 - n^2$; $z = m^2 + n^2$

   where m, n are natural numbers with $m > n$.

   The most significant thing about this formula is that it generates ALL Pythagorean triples.

4. What is considered to be the Pythagorean School's greatest contribution to Mathematics?

   The discovery of irrational numbers. In particular, the proof that the side and diagonal of a square are "incommensurable" (i.e., not of "common measure") , or equivalently, that $\sqrt{2}$ is irrational.

5. What did the Pythagoreans mean when they defined two line segments to have "**common measure**"?

   Two line segments $\overline{AB}$ and $\overline{CD}$ are **of common measure** if there exists a third line segment of length δ and natural numbers m and n such that

   $$\left|\overline{AB}\right| = m\delta$$

   and

   $$\left|\overline{CD}\right| = n\delta.$$

Equivalently: line segments $\overline{AB}$ and $\overline{CD}$ are **of common measure** if

$$\frac{|\overline{AB}|}{|\overline{CD}|} = \frac{m\delta}{n\delta} = \frac{m}{n}.$$

The ratio of the lengths of line segments $\overline{AB}$ and $\overline{CD}$ is the quotient of integers (a rational number).

Alternatively: line segments $\overline{AB}$ and $\overline{CD}$ are **of common measure** if there exists a third line segment of length δ such that the line segment of length δ can be marked off a whole number of times on each of the line segments $\overline{AB}$ and $\overline{CD}$.

6. Another "proof" of the irrationality of $\sqrt{2}$ is a "construction proof" involving a unit square and the construction of an infinite sequence of successively smaller squares, using this unit square. How does this construction yield the contradiction that turns out to be the "crux" of the proof? (Give the basic idea. You don't have to go into rigorous details.)

   By constructing an infinite sequence of squares whose sides are supposedly commensurable with their diagonals, we produce an infinite, strictly-decreasing sequence of natural numbers (an impossibility). Since the assumption that the side and diagonal of a square are commensurable leads to a contradiction, the assumption must be false. Hence, the side and diagonal of a square are NOT commensurable.

7. What contributions to mathematics were made by Theodorus of Cyrene and Theaetetus of Athens?

   Theodorus proved that squares with sides of length $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, and $\sqrt{17}$ are incommensurable with unit length.

   Equivalently: Theodorus proved that the square roots of integers from 3 to 17, that are non-perfect squares, are irrational.

   Theaetetus (a pupil of Theodorus) extended the result to the square roots of all non-perfect square integers.

8. Name three famous construction problems of the Greeks, and briefly describe what each problem calls for.

   The following three "construction" problems were to be solved such that:

   1. a straightedge could be used to draw a line through two given points

   and

   2. a compass could be used to draw a circle with given center and radius

Problem 1: (The "Squaring of a Circle") Given a circle, construct a square whose area is equal to the circle.

Problem 2: (The "Doubling of a Cube" or the "Duplication of a Cube") Given a cube, construct a cube whose volume is twice that of the given cube. (i.e., finding the edge of a cube having twice the volume of a given cube.)

Problem 3: The Trisection of a General Angle

9. How were the constructions in these problems to be performed?

   The aforementioned three "construction" problems were to be solved such that:

   1. a straightedge could be used to draw a line through two given points

   and

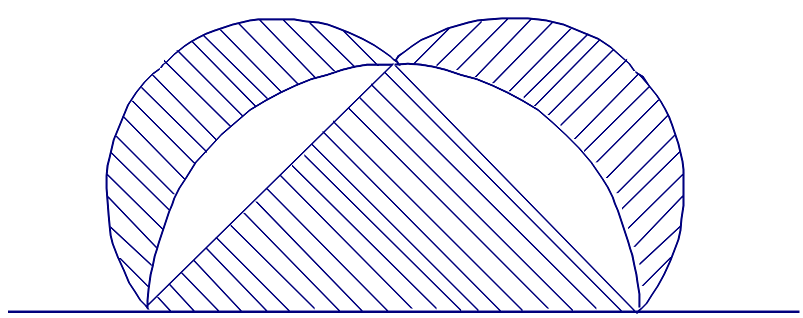
   2. a compass could be used to draw a circle with given center and radius

10. What noteworthy construction problem did Hippocrates (of Chios) solve, and what is so unusual about this problem?

    Hippocrates constructed a pair of lunes (moon-shaped figures bounded by circular arcs of unequal radii) whose combined area was equal to that of a right, isosceles triangle. (See below)

    Equivalently, he constructed two pair of lunes whose combined area was equal to that of a square.

    In so doing, Hippocrates was able to show that there are *some* plane regions with curved boundaries that *can* be "squared." (i.e., There are some regions with curvilinear boundaries such that given such a region, it *is* possible to construct a square having the same area.)



11. What is considered to be Hippocrates' most noteworthy contribution to mathematics?

In his attempt to "Double a Cube," Hippocrates reduced a problem in solid geometry (The "Doubling of a Cube") to a simpler problem in plane geometry (finding two mean proportionals between a given line segment and another line segment twice its length. i.e., given line segments of lengths a and $2a$, construct two line segments of length x and y, such that $\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$.)