Solutions to p. 117 Exercises

Summer 2024

Pat Rossi Name ______________________________

the exercises #1, 4, 15, 16, 17

1.a. Establish the formula:

$$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$

Observe:

$ab + \left(\frac{a-b}{2}\right)^2 = ab + \frac{(a-b)^2}{2^2} = ab + \frac{a^2-2ab+b^2}{4} = \frac{4ab}{4} + \frac{a^2-2ab+b^2}{4} = \frac{a^2+2ab+b^2}{4}$

$= \frac{(a+b)^2}{2^2} = \left(\frac{a+b}{2}\right)^2$

i.e., $ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$

1.b. Show that $a = 2n^2$, $b = 2$, gives rise to Plato's formula to Pythagorean triples, whereas $a = (2n+1)^2$, $b = 1$, yields Pythagoras' own formula.

Plato's formula: $x = 2n$, $y = n^2 - 1$, $z = n^2 + 1$

Given: $ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$ and letting $a = 2n^2$, $b = 2$; we have:

$(2n^2)(2) + \left(\frac{2n^2-2}{2}\right)^2 = \left(\frac{2n^2+2}{2}\right)^2$

Same as: $(2n)^2 + (n^2-1)^2 = (n^2+1)^2$

This is **Plato's formula:** $x = 2n$, $y = n^2 - 1$, $z = n^2 + 1$

$$\boxed{\textbf{Pythagoras' formula: } x = 2n+1,\ y = 2n^2+2n,\ z = 2n^2+2n+1}$$

Given: $ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$ and letting $a = (2n+1)^2$, $b = 1$; we have:

$$(2n+1)^2 \cdot 1 + \left(\frac{(2n+1)^2 - 1}{2}\right)^2 = \left(\frac{(2n+1)^2 + 1}{2}\right)^2$$

Same as: $(2n+1)^2 + \left(\frac{\left(4n^2+4n+1\right)-1}{2}\right)^2 = \left(\frac{\left(4n^2+4n+1\right)+1}{2}\right)^2$

Same as: $(2n+1)^2 + \left(\frac{4n^2+4n}{2}\right)^2 = \left(\frac{4n^2+4n+2}{2}\right)^2$

Same as: $(2n+1)^2 + \left(2n^2+2n\right)^2 = \left(2n^2+2n+1\right)^2$

$$\boxed{\text{This is } \textbf{Pythagoras' formula: } x = 2n+1,\ y = 2n^2+2n,\ z = 2n^2+2n+1}$$

4. Verify that $(3, 4, 5)$ is the only Pythagorean triple involving consecutive integers.

   pf/ Suppose that $x, x+1$, and $x+2$ form a Pythagorean triple.

   Then $x^2 + (x+1)^2 = (x+2)^2$

   $\Rightarrow x^2 + \left(x^2 + 2x + 1\right) = x^2 + 4x + 4$

   $\Rightarrow 2x^2 + 2x + 1 = x^2 + 4x + 4$

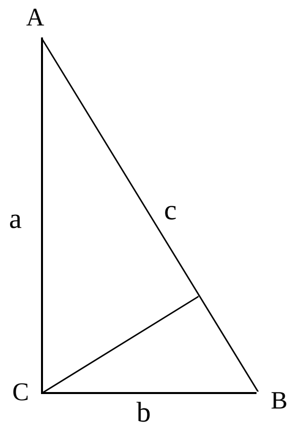
   $\Rightarrow x^2 - 2x - 3 = 0$

   $\Rightarrow (x+1)(x-3) = 0$

   $\Rightarrow x = -1$, or $x = 3$

   Since x is part of a Pythagorean Triple, $x > 0$.
   Hence, $x = 3$ and consequently, $(x, x+1, x+2) = (3, 4, 5)$ is the only Pythagorean triple made up of consecutuive integers.

15. A standard proof of the Pythagoeran Theorem starts with a right triangle ABC, with its right angle at C, and then draws a perpendicular CD from C to the hypotenuse AB.



1. Show that Triangles ACD and CBD are both similar to triangle ABC.

Triangle ACD is similar to triangle ABC

Triangle ACD and triangle ABC have a common angle (angle A).

$\angle ADC$ of Triangle ACD is a right angle, as is $\angle ACB$ of triangle ABC.

Since Triangle ACD and triangle ABC have two congruent angles, the remaining angle in each triangle must also be congruent to one another.

Hence, triangles ACD and triangle ABC are similar.

Triangle CBD is similar to triangle ABC

Triangle CBD and triangle ABC have a common angle (angle B).

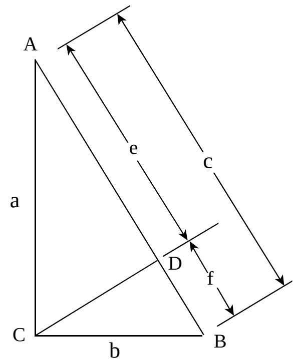
$\angle BDC$ of Triangle CBD is a right angle as is $\angle ACB$ of triangle ABC.

Since Triangle CBD and triangle ABC have two congruent angles, the remaining angle in each triangle must also be congruent.

Hence, triangles CBD and triangle ABC are similar.

2. For triangle ABC with legs of lengths a and b, and hypotenuse of length c, use the proportionality of corresponding sides of similar triangles to establish that $a^2 + b^2 = c^2$.

To do this proof, let's label some unlabeled sides in the previous diagram.



Using the proportional property of similar triangles (ABC similar to ACD), we have:

$\frac{c}{a} = \frac{a}{e}$

$\Rightarrow ce = a^2$

i.e., $a^2 = ce \qquad \text{(eq. 1)}$

Using the proportional property of similar triangles (ABC similar to CBD), we have:

$\frac{b}{f} = \frac{c}{b}$

$\Rightarrow b^2 = cf \qquad \text{(eq. 2)}$

Thus, from eq. 1 and eq. 2, we have:

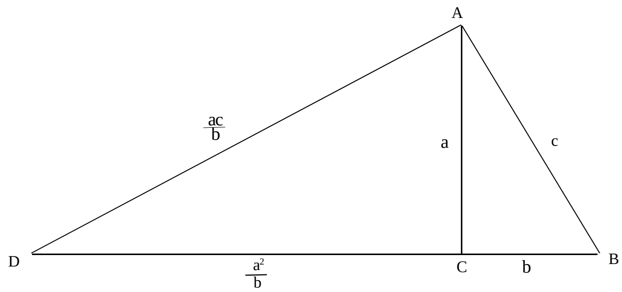
$a^2 + b^2 = ce + cf = c\,(e + f) = c \cdot c = c^2$

i.e., $a^2 + b^2 = c^2$

16. For another proof of the Pythagorean Theorem, consider a right traingle ABC (with right angle at C) whose legs have length a and b and whose hypotenuse has length c. On th extension of side BC pick a point D such that BAD is a right triangle.

   a. From the similarity of triangles ABC and DBA, show that $AD = \frac{ac}{b}$ and $DC = \frac{a^2}{b}$.

   b. Prove that $a^2 + b^2 = c^2$ by relating the area of triangles ABC and ACD.



a. From the similarity of triangles ABC and DBA, show that $AD = \frac{ac}{b}$ and $DC = \frac{a^2}{b}$.

Comparing corresponding sides of similar triangles, we have:
$\frac{\overline{AD}}{c} = \frac{a}{b} \Rightarrow \overline{AD} = \frac{ac}{b}$

Also:

$\frac{\overline{DC}}{a} = \frac{a}{b} \Rightarrow \overline{DC} = \frac{a^2}{b}$

b. Prove that $a^2 + b^2 = c^2$ by relating the area of triangles ABC and ACD.

Observe:

Area $ABC = \frac{1}{2}ab$

Area $ACD = \frac{1}{2}\frac{a^2}{b} \cdot a$

Area $ABD = \frac{1}{2}\frac{ac}{b} \cdot c$

Observe that: Area $ABC+$ Area $ACD =$ Area ABD

i.e., $\frac{1}{2}ab + \frac{1}{2}\frac{a^2}{b} \cdot a = \frac{1}{2}\frac{ac}{b} \cdot c$

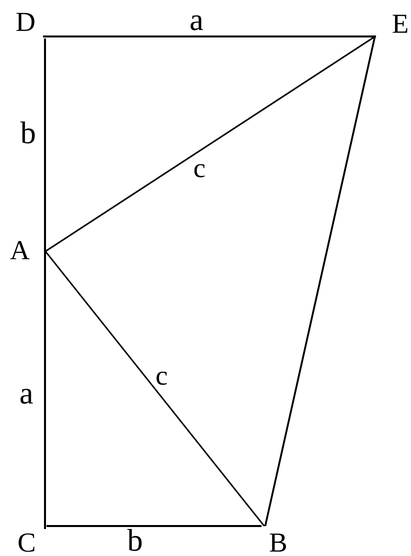
$\Rightarrow ab + \frac{a^2}{b} \cdot a = \frac{ac}{b} \cdot c$

$\Rightarrow ab^2 + a^2 \cdot a = ac \cdot c$

$\Rightarrow b^2 + a^2 = c \cdot c$

i.e. $a^2 + b^2 = c^2$

17. Several years before James Garfield became president of the United States, he devised an original proof of the Pythagorean Theorem. It appeared in 1876 in the New England Journal of Education. Starting with a right triangle ABC, Garfield placed an Congruent triangle EAD as indicated in the figure. He then drew EB so as to form a quadrilateral $EBCD$. Prove that $a^2 + b^2 = c^2$ by relating the area of the quadrilateral to the area of the three triangles ABC, EAD, and EBA.



First, note that $\angle EAB$ is a right triangle. Here's why:

$\angle DEA + \angle DAE = 90\,^\circ$, since these angles are the complementary angles in the right triangle EDA.

Also:

$\angle CAB + \angle CBA = 90\,^\circ$, since these angles are the complementary angles in the right triangle ACB.

Finally note that: $\angle DAE \cong \angle CAB$, being corresponding angles of congruent triangles.

Hence, $\angle DEA + \angle DAE = 90\,^\circ \Rightarrow \angle DEA + \angle CAB = 90\,^\circ$

i.e., $\angle DEA$ and $\angle CAB$ are complementary angles.

Hence, since $\angle DEA + \angle EAB + \angle CAB = 180\,^\circ$, $\angle EAB = 90\,^\circ$.

i.e., $\angle EAB$ is a right angle.

Observe:

Area EAD $= \frac{1}{2}ab$

Area EBA $= \frac{1}{2}c \cdot c = \frac{1}{2}c^2$

Area ACB $= \frac{1}{2}ab$

Area quadrilateral $EBCD = \frac{a+b}{2} \cdot (a+b) = \frac{(a+b)^2}{2}$

Equating the area of quadrilateral $EBCD$ to the area of the three triangles ABC, EAD, and EBA, we have:

$\frac{(a+b)^2}{2} = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$

$\Rightarrow (a+b)^2 = ab + c^2 + ab$

$\Rightarrow a^2 + 2ab + b^2 = 2ab + c^2$

$\Rightarrow a^2 + b^2 = c^2$