P. 80 - Exercises and Solutions

Term 5 2024

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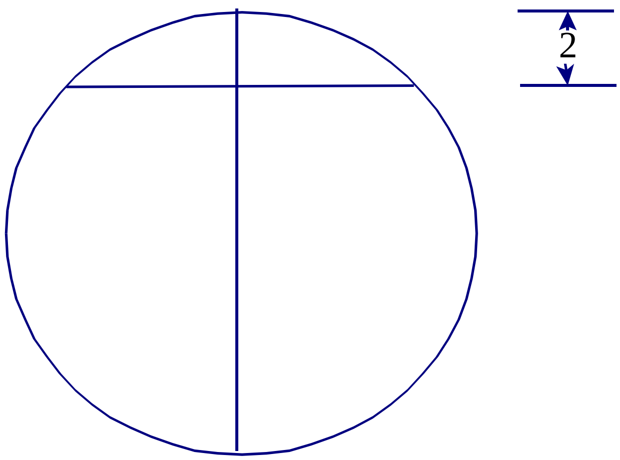
Ex 1 Given that the circumference of a circle is 60 units and the length of a perpendicular from the center of a chord of the circle to the circumference is 2 units, find the length of the chord. In solving the problem, use $\pi = 3$, as did the Babylonians.



We are given that the Circumference is 60

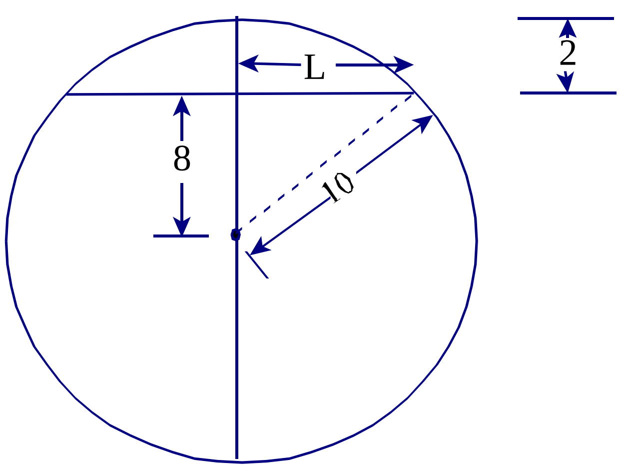
i.e., $C = 2\pi r = 60$

Using $\pi = 3$, this yields:

$C = 6r = 60$

$\Rightarrow r = 10$

If we let the length of the chord be $2L$, then we have the following:



By Pythagorean's Theorem: $L^2 + 8^2 = 10^2$

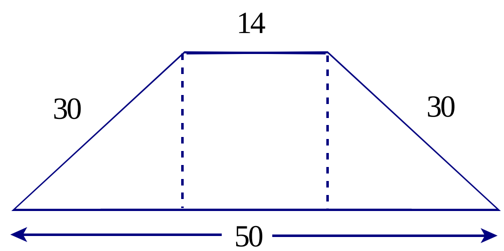
$\Rightarrow L^2 = 10^2 - 8^2$

$\Rightarrow L = \sqrt{36} = 6$

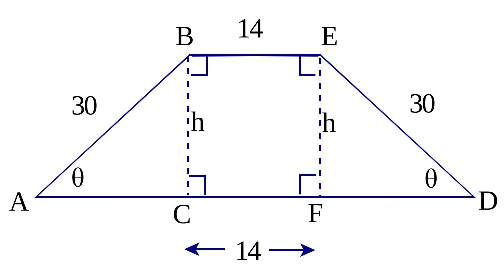
We are asked to find the length of the chord, which has length $2L$.

Thus, the length of the chord is $2L = 12$

Ex 2 - An Old Babylonian tablet calls for finding the area of an isosceles trapezoid whose sides are 30 units long and whose bases are 14 and 50 units. Solve this problem. (Note: an isosceles trapezoid is a trapezoid whose non-parallel sides are equal and whose base angles are equal.)

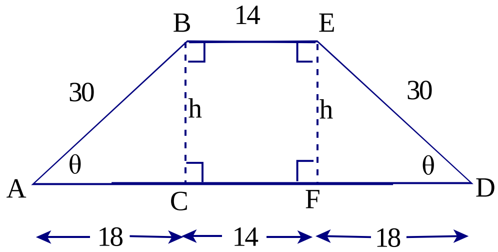


The top and base of the trapezoid are parallel. Hence the vertical sides of the triangles are equal, since they each represent the distance from the base to the top of the trapezoid. We'll call this distance h. The base angles of the trapezoid are equal by definition of "isosceles trapezoid." We'll denote these by θ. (See below.)



The remaining angle in each triangle is equal to $180^\circ - (90^\circ + \theta)$. ($180^\circ$ minus the measures of the other two angles.) Hence, the triangles are similar. As a consequence, the measures of corresponding sides of the triangle are proportional. Since the vertical side of each triangle is h, the proportion is 1. (i.e., the triangles are congruent.)

As another consequence, the length of line segment $\overline{AC}$ is equal to the length of line segment $\overline{DF}$. Hence, each has the length $\frac{50-14}{2} = 18$. (See below.)



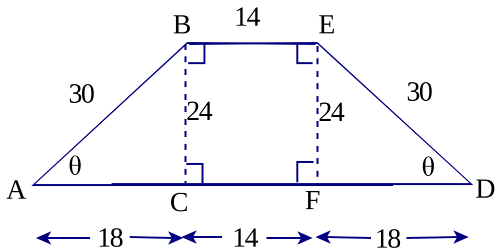
By Pythagorean's Theorem:

$h^2 + 18^2 = 30^2$

$\Rightarrow h^2 = 30^2 - 18^2$

$\Rightarrow h^2 = 576$

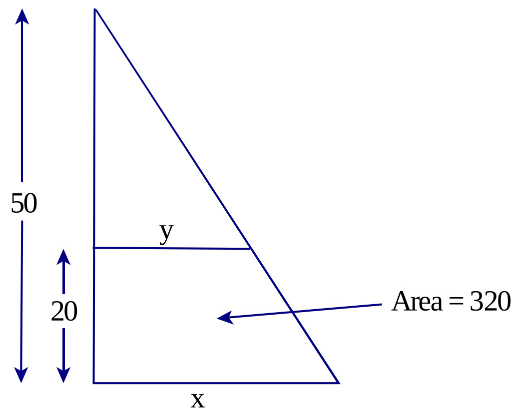
$\Rightarrow h = 24$



The area of the trapezoid is equal to the sum of the areas of the two triangles plus the area of the central rectangle.

$\Rightarrow A = \frac{1}{2}(18)(24) + \frac{1}{2}(18)(24) + (14)(24) = 768 \text{ units}^2$

Ex 3 - In another tablet, one side of a right triangle is 50 units long. Parallel to the other side, and 20 units from this other side, a line is drawn that cuts off a right trapezoid of area 320 units^2. (See the diagram) Find the lengths of the bases (i.e., the parallel sides) of the trapezoid. (Hint: If A is the area of the "original triangle" (i.e. the large triangle), then $320 + 15y = A = 25x$, and $\frac{1}{2}(x+y)\,20 = 320$.)



First, observe that the area of the "bottom triangle" (i.e., the "large triangle") is given by:

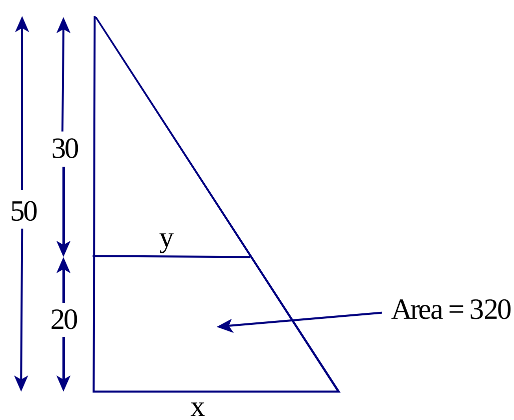
$A_{\text{large}} = \frac{1}{2}bh = \frac{1}{2}x \cdot 50 = 25x$

i.e., the area of the "large triangle" is $A_{\text{large triangle}} = 25x$ (Eq. 1)

Next, observe that the area of the trapezoid is given by the average of the lengths of the parallel sides times the height

i.e. The area of the trapezoid is given by $A_{\text{trap}} = \frac{1}{2}(x+y) \cdot 20 = 320$

i.e. The area of the trapezoid is given by $A_{\text{trap}} = 10x + 10y = 320$ (Eq. 2)



The area of the "top triangle" is given by $A_{\text{top triangle}} = \frac{1}{2}bh = \frac{1}{2}y \cdot 30 = 15y$

i.e., the area of the "top triangle" is $A_{\text{top triangle}} = 15y$ (Eq. 3)

Finally, observe that the area of the "large triangle" is equal to the sum of the areas of the trapezoid and the "top triangle" i.e., the area of the "large triangle" is given by $A_{\text{large triangle}} = \underbrace{320}_{A_{\text{trap}}} + \underbrace{15y}_{A_{\text{top triangle}}}$ (Eq. 4)

From Eq. 1 and Eq. 4, we have: $A_{\text{large}} = 25x = 320 + 15y$

i.e., $25x = 320 + 15y$

$\Rightarrow 25x - 15y = 320$ (Eq. 5)

Eq. 2 and Eq. 5 give us a system of equations that we can solve of x and y

$10x + 10y = 320$

$25x - 15y = 320$

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$\Rightarrow 30x + 30y = 960$

$\Rightarrow 50x - 30y = 640$

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$\Rightarrow 80x \qquad = 1600$

$\Rightarrow x = 20$
$\Rightarrow y = 12$

Recall: We were asked to "Find the lengths of the bases (i.e., the parallel sides) of the trapezoid."

"The lengths of the bases of the trapezoid" are: $x = 20$ and $y = 12$

Ex 4 A triangle whose base has length 30 is divided into two parts by a line segment drawn parallel to its base. It is given that the resulting right trapezoid has an area larger by 7,0 = 420 than the upper triangle, and that the difference between the height y of the upper triangle and the height z of the trapezoid is 20. If x is the length of the upper base of the trapezoid these statements lead to the equations:

$$\frac{1}{2}z\,(x+30) = \frac{1}{2}xy + 420; \qquad y - z = 20$$

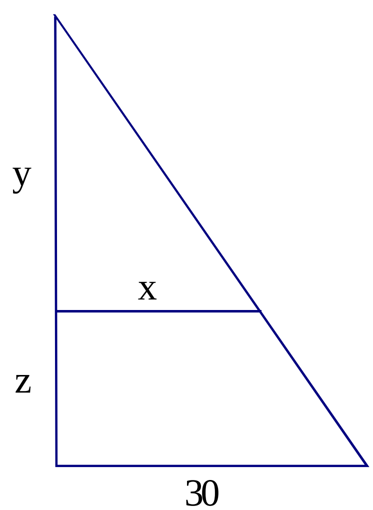
Find the quantities x, y, z using equivalent triangles:

$\frac{y}{y+z} = \frac{x}{30}$ or equivalently: $\frac{y+z}{30} = \frac{y}{x}$

WHAT???

Let's take this "piece by piece."

We're given a "labeled diagram" of the situation. (below)



We're told that: "the resulting right trapezoid has an area larger by 7,0 = 420 than the upper triangle"

i.e., $A_{\text{trap}} = A_{\text{upper Triangle}} + 420$

i.e., (avg of parallel sides) $\cdot$ (height) $= \frac{1}{2}$ (base) (height) + 420

i.e., $\left(\frac{x+30}{2}\right) \cdot z = \frac{1}{2}x \cdot y + 420$

$\Rightarrow (x+30) \cdot z = x \cdot y + 420$ (eq. 1)

Next: We're told that: "the difference between the height y of the upper triangle and the height z of the trapezoid is 20."

i.e., $y - z = 20$

From similar triangles, we have:

$\frac{y+z}{30} = \frac{y}{x}$

$\Rightarrow x = \frac{30y}{y+z}$ (eq. 2)

From the equation $y - z = 20$, we have:

$z = y - 20 \qquad \text{(eq. 3)}$

Substitute this for z in the eq. 2, and this yields:

$x = \frac{30y}{y+(y-20)}$

$\Rightarrow x = \frac{30y}{2y-20}$

$\Rightarrow x = \frac{15y}{y-10} \qquad \text{(eq. 4)}$

Recall: $(x+30) \cdot z = x \cdot y + 420 \qquad \text{(eq. 1)}$

From eq. 3, $z = y - 20$. So we can substitute $y - 20$ for z in the previous equation, yielding:

$\Rightarrow (x+30) \cdot (y-20) = xy + 420$

$\Rightarrow (x+30)(y-20) = xy + 840$

$\Rightarrow xy + 30y - 20x - 600 = xy + 840$

$\Rightarrow 30y - 20x = 1440$

From eq. 4, we can substitute $\frac{15y}{y-10}$ for x in the previous equation, yielding:

$30y - 20\frac{15y}{y-10} = 1440$

$\Rightarrow 30y(y-10) - 20(15y) = 1440(y-10)$

$\Rightarrow 30y^2 - 300y - 300y = 1440y - 14400$

$\Rightarrow 30y^2 - 2040y + 14400 = 0$

$\Rightarrow y^2 - 68y + 480 = 0$

$\Rightarrow (y-8)(y-60) = 0$

$\Rightarrow y = 8; \qquad y = 60$

Note: since $z = y - 20$ must be a positive number, it follows that $y > 20$.

Hence, we discard the solution $y = 8$.

$\Rightarrow y = 60$

$\Rightarrow z = y - 20 = 40$

and $x = \frac{15y}{y-10} = \frac{15(60)}{(60)-10} = 18$

i.e., $y = 60; \qquad z = 40; \qquad x = 18$