MTH 1126 – Test #1 – Solutions – 9am Class

SPRING 2022

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Show CLEARLY how you arrive at your answers

1. Compute: $\frac{d}{dx}\left[e^{\sin\left(2x^2\right)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\sin\left(2x^2\right)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\sin\left(2x^2\right)}}_{e^u} \cdot \underbrace{\frac{d}{dx}\left[\sin\left(2x^2\right)\right]}_{\frac{du}{dx}} = e^{\sin\left(2x^2\right)} \cdot \cos\left(2x^2\right) \cdot 4x$$

i.e., $\frac{d}{dx}\left[e^{\sin\left(2x^2\right)}\right] = e^{\sin\left(2x^2\right)} \cdot \cos\left(2x^2\right) \cdot 4x$

Or: $\frac{d}{dx}\left[e^{\sin\left(2x^2\right)}\right] = 4x\cos\left(2x^2\right) e^{\sin\left(2x^2\right)}$

2. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{\sin(x)}{x^3}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{\sin(x)}{x^3}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(\sin\left(x\right)\right) - \ln\left(x^3\right)\right)\right]$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{\sin\left(x\right)}}_{\frac{i}{u}} \cdot \underbrace{\cos\left(x\right)}_{\frac{du}{dx}} - \underbrace{\frac{1}{x^3}}_{\frac{i}{u}} \cdot \underbrace{3x^2}_{\frac{du}{dx}}\right) = \frac{1}{2}\left(\frac{\cos(x)}{\sin(x)} - \frac{3x^2}{x^3}\right) = \frac{1}{2}\left(\cot\left(x\right) - \frac{3}{x}\right) = \frac{1}{2}\cot\left(x\right) - \frac{3}{2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] = \frac{1}{2}\cot\left(x\right) - \frac{3}{2x}$

(Alternative Solution Appears on the Following Page)

Alternative Solution:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{\sin(x)}{x^3}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{\sin(x)}{x^3}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{\sin(x)}{x^3}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\frac{\sin(x)}{x^3}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{\sin(x)}{x^3}\right)^{-\frac{1}{2}} \underbrace{\frac{(\cos(x))(x^3) - (3x^2)(\sin(x))}{(x^3)^2}}_{\text{Quotient Rule}}$$

$$= \left(\frac{x^3}{\sin(x)}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{x^3}{\sin(x)}\right)^{\frac{1}{2}} \frac{(\cos(x))(x^3) - (3x^2)(\sin(x))}{x^6}$$

$$= \frac{1}{2}\left(\frac{x^3}{\sin(x)}\right) \frac{(\cos(x))(x^3) - (3x^2)(\sin(x))}{x^6} = \frac{x^6\cos(x) - 3x^5\sin(x)}{2x^6\sin(x)} = \frac{x^6\cos(x)}{2x^6\sin(x)} - \frac{3x^5\sin(x)}{2x^6\sin(x)} = \frac{1}{2}\cot(x) - \frac{3}{2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] = \frac{1}{2}\left(\frac{x^3}{\sin(x)}\right) \frac{(\cos(x))(x^3) - (3x^2)(\sin(x))}{x^6}$

Or: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\sin(x)}{x^3}}\right)\right] = \frac{1}{2}\cot(x) - \frac{3}{2x}$

3. Compute: $\int e^{\left(7x^4+4x^3\right)}\left(7x^3+3x^2\right)\,dx =$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $e^{\left(7x^4+4x^3\right)}$

Let $u = 7x^4 + 4x^3$

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(7x^4+4x^3\right)}_{\text{function}} \rightarrow \underbrace{\left(7x^3+3x^2\right)}_{\text{deriv}}$

Let $u = (7x^4 + 4x^3)$

2. Compute du

$$\begin{array}{|rrcl|}\hline & u & = & 7x^4+4x^3 \\ \Rightarrow & \frac{du}{dx} & = & 28x^3+12x^2 \\ \Rightarrow & du & = & \left(28x^3+12x^2\right)dx \\ \Rightarrow & \frac{1}{4}du & = & \left(7x^3+3x^2\right)dx \\ \hline\end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(7x^4+4x^3\right)}}_{e^u}\underbrace{\left(7x^3+3x^2\right)\,dx}_{\frac{1}{4}du} = \int e^u \tfrac{1}{4}du = \tfrac{1}{4}\int e^u\,du$$

4. Integrate in terms of u

$\frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C$

5. Re-write in terms of x

$$\int e^{\left(7x^4+4x^3\right)}\left(7x^3+3x^2\right)\,dx = \underbrace{\frac{1}{4}e^{\left(7x^4+4x^3\right)} + C}_{\frac{1}{4}e^u+C}$$

$$\boxed{\text{i.e., } \int e^{\left(7x^4+4x^3\right)}\left(7x^3+3x^2\right)\,dx = \tfrac{1}{4}e^{\left(7x^4+4x^3\right)} + C}$$

4. Compute: $\int \frac{3x-1}{(3x^2-2x+5)^3}\,dx = \int \frac{1}{(3x^2-2x+5)^3}\,(3x-1)\,dx = \int \left(3x^2-2x+5\right)^{-3}(3x-1)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\left(3x^2-2x+5\right)^{-3}$

   Let $u = (3x^2-2x+5)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\left(3x^2-2x+5\right)}_{\text{function}} \to \underbrace{6x-2}_{\text{deriv}}$

   Let $u = (3x^2-2x+5)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 3x^2-2x+5 \\ \Rightarrow & \frac{du}{dx} & = & 6x-2 \\ \Rightarrow & du & = & (6x-2)\,dx \\ \Rightarrow & \frac{1}{2}du & = & (3x-1)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\left(3x^2-2x+5\right)^{-3}}_{u^{-3}} \underbrace{(3x-1)\,dx}_{\frac{1}{2}du} = \int u^{-3}\tfrac{1}{2}du = \tfrac{1}{2}\int u^{-3}\,du$$

4. Integrate in terms of u

$$\tfrac{1}{2}\int u^{-3}\,du = \tfrac{1}{2}\tfrac{u^{-2}}{-2} + C = -\tfrac{1}{4}u^{-2} + C$$

5. Re-write in terms of x

$$\int \frac{3x-1}{(3x^2-2x+5)^3}\,dx = \underbrace{-\frac{1}{4}\left(3x^2-2x+5\right)^{-2} + C}_{-\frac{1}{4}u^{-2}+C}$$

$$\boxed{\text{i.e., } \int \frac{3x-1}{(3x^2-2x+5)^3}\,dx = -\tfrac{1}{4}\left(3x^2-2x+5\right)^{-2} + C}$$

5. Compute: $\int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \int \frac{1}{(x^3+3x^2+3x)}\,(x^2+2x+1)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{(x^3+3x^2+3x)} = (x^3+3x^2+3x)^{-1}$

   Let $u = (x^3+3x^2+3x)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{(x^3+3x^2+3x)}_{\text{function}} \to \underbrace{(x^2+2x+1)}_{\text{deriv}}$

   Let $u = (x^3+3x^2+3x)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\begin{array}{|rrcl|}\hline & u & = & x^3+3x^2+3x \\ \Rightarrow & \frac{du}{dx} & = & 3x^2+6x+3 \\ \Rightarrow & du & = & (3x^2+6x+3)\,dx \\ \Rightarrow & \frac{1}{3}du & = & (x^2+2x+1)\,dx \\ \hline\end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{(x^3+3x^2+3x)}}_{\frac{1}{u}} \underbrace{(x^2+2x+1)\,dx}_{\frac{1}{3}du} = \int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \underbrace{\frac{1}{3}\ln\left|x^3+3x^2+3x\right| + C}_{\frac{1}{3}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \frac{1}{3}\ln|x^3+3x^2+3x| + C}$$

6. Compute: $\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right] =$

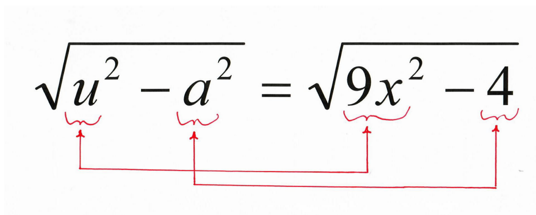
$$\underbrace{\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right]}_{\frac{d}{dx}[\arcsin(u)]} = \underbrace{\frac{1}{\sqrt{1-\left(\tan\left(x\right)\right)^2}}}_{\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{\sec^2\left(x\right)}_{\frac{du}{dx}} = \frac{\sec^2(x)}{\sqrt{1-(\tan(x))^2}} = \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}$$

i.e., $\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right] = \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}$

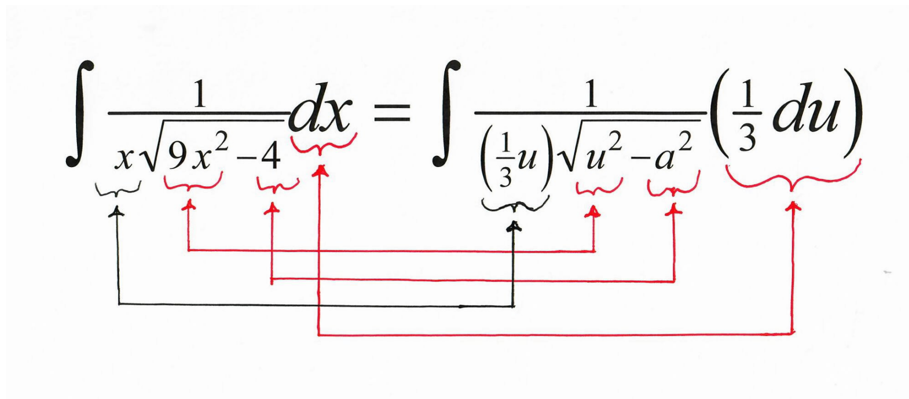
7. Compute: $\int \frac{1}{x\sqrt{9x^2-4}}\,dx$

This appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

If our conjecture is correct, then $\sqrt{u^2-a^2} = \sqrt{9x^2-4}$



$$\begin{aligned} \Rightarrow\ & a^2 = 4 \\ & a = 2 \\ \Rightarrow\ & u^2 = 9x^2 \\ & u = 3x \\ & \tfrac{1}{3}u = x \\ \Rightarrow\ & \tfrac{du}{dx} = 3 \\ & du = 3dx \\ & \tfrac{1}{3}du = dx \end{aligned}$$



3. Analyze in terms of u and du.

$$\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \int \frac{1}{x\sqrt{(3x)^2-2^2}}dx = \int \frac{1}{\left(\frac{1}{3}u\right)\sqrt{u^2-a^2}}\left(\tfrac{1}{3}du\right) = \int \frac{1}{u\sqrt{u^2-a^2}}du$$

4. Integrate

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

5. Re-express in terms of x

$$\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \underbrace{\frac{1}{2}\operatorname{arcsec}\left(\frac{|3x|}{2}\right) + C}_{\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C}$$

$$\boxed{\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \frac{1}{2}\operatorname{arcsec}\left(\frac{|3x|}{2}\right) + C}$$

8. Compute: $\frac{d}{dx}\left[\tan^{-1}\left(\sqrt{x}\right)\right] = \frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{1}{2}}\right)\right]$

$$\underbrace{\frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{1}{2}}\right)\right]}_{\frac{d}{dx}[\tan^{-1}(u)]} = \underbrace{\frac{1}{1+\left(x^{\frac{1}{2}}\right)^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{\frac{1}{2}x^{-\frac{1}{2}}}_{\frac{du}{dx}} = \frac{1}{(1+x)}\frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2x^{\frac{1}{2}}+2x^{\frac{3}{2}}} = \frac{1}{2x^{\frac{1}{2}}+2x^{\frac{3}{2}}}\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{2x+2x^2}$$

i.e., $\frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{1}{2}}\right)\right] = \frac{1}{2x^{\frac{1}{2}}+2x^{\frac{3}{2}}} = \frac{x^{\frac{1}{2}}}{2x+2x^2}$

9. Compute: $\int \frac{1}{\sqrt{9-16x^4}} x\,dx = \int \frac{1}{(9-16x^4)^{\frac{1}{2}}} x\,dx = \int \left(9-16x^4\right)^{-\frac{1}{2}} x\,dx$

1. a. Is there a composite function?

   Yes. $\underbrace{(9-16x^4)}_{\text{inner}}{}^{-\frac{1}{2}}$

   Let $u = 9 - 16x^4$

   Is there an "approximate function/derivative pair"?

   There does not appear to be an "approximate function/derivative pair."

   Proceeding solely on the strength of Part a, we continue, aware of the possibility that u-substitution might not work.

2. Compute du

$$\begin{array}{|rrcl|} & u & = & 9-16x^4 \\ \Rightarrow & \frac{du}{dx} & = & -64x^3 \\ \Rightarrow & du & = & -64x^3dx \\ \Rightarrow & -\frac{1}{64x^2}du & = & x\,dx \end{array}$$

3. Analyze in terms of u and du.

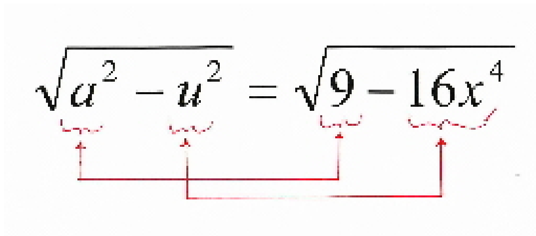
$$\int \underbrace{\left(9-16x^4\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \cdot \underbrace{x\,dx}_{\frac{1}{-64x^2}du} =$$

   Since we cannot analyze the integral solely in terms of u and du, u-substitution alone will not work in this case.

   We must try to get our integral to fit a different form. (See Next Page)

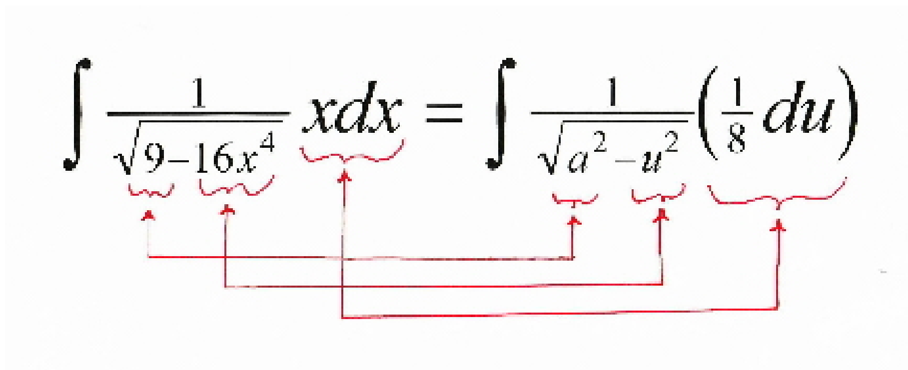
Exercise 9 Continued . . .

$\int \frac{1}{\sqrt{9-16x^4}} x\,dx$ compare to: $\int \frac{1}{\sqrt{a^2-u^2}} du$



If this comparison is correct, then:

	$a^2 = 9$
$\Rightarrow$	$a = 3$
	$u^2 = 16x^4$
$\Rightarrow$	$u = 4x^2$
$\Rightarrow$	$\frac{du}{dx} = 8x$
$\Rightarrow$	$du = 8xdx$
$\Rightarrow$	$\frac{1}{8}du = xdx$



Now analyze the integral in terms of u and du.

$$\int \frac{1}{\sqrt{9-16x^4}} x\,dx = \int \frac{1}{\sqrt{3^2-(4x^2)^2}} x\,dx = \int \frac{1}{\sqrt{a^2-u^2}} \frac{1}{8} du = \frac{1}{8} \int \frac{1}{\sqrt{a^2-u^2}} du$$

3. Integrate:

$$\frac{1}{8} \int \frac{1}{\sqrt{a^2-u^2}} du = \frac{1}{8} \arcsin\left(\frac{u}{a}\right) + C = \frac{1}{8} \arcsin\left(\frac{4x^2}{3}\right) + C$$

i.e., $\int \frac{1}{\sqrt{9-16x^4}} x\,dx = \frac{1}{8} \arcsin\left(\frac{4x^2}{3}\right) + C$

10. $z = \tan\left(\arccos\left(\frac{x-1}{2}\right)\right)$ Re-write this equation as an equivalent algebraic equation.

Let $w = \arccos\left(\frac{x-1}{2}\right)$

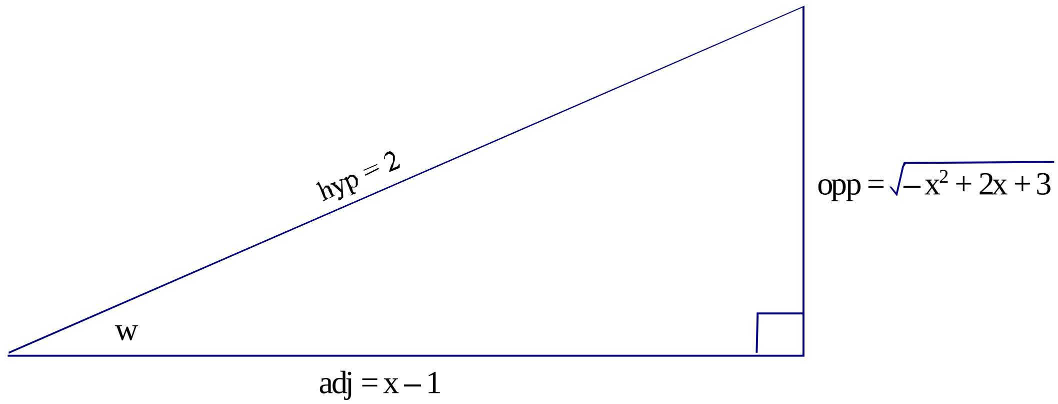
Then "w is the angle whose cosine is $\frac{x-1}{2}$."

i.e., $\cos(w) = \frac{x-1}{2}$

Draw a right triangle that depicts this relationship.

i.e., $\cos(w) = \frac{x-1}{2} = \frac{\text{adj}}{\text{hyp}} = \frac{x-1}{2}$

$opp^2 = hyp^2 - adj^2 = 2^2 - (x-1)^2 = 4 - (x^2 - 2x + 1) = -x^2 + 2x + 3$
i.e., $opp^2 = -x^2 + 2x + 3$
$\Rightarrow opp = \sqrt{-x^2 + 2x + 3}$



We want $z = \tan\left(\arccos\left(\frac{x-1}{2}\right)\right)$

But since $w = \arccos\left(\frac{x-1}{2}\right)$,

$\Rightarrow z = \tan(w)$

$\Rightarrow z = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{-x^2+2x+3}}{x-1}$

i.e., $z = \frac{\sqrt{-x^2+2x+3}}{x-1}$

Extra: Wow! 10 points (All or nothing)

Compute: $\int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)} dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{\cos^2(x)-\sin^2(x)} = \left(\cos^2(x) - \sin^2(x)\right)^{-1}$

   Let $u = \cos^2(x) - \sin^2(x)$ i.e., "Let u = the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\cos^2(x) - \sin^2(x)}_{\text{function}} \to \underbrace{\sin(x)\cos(x)}_{\text{deriv}}$

   Let $u = \cos^2(x) - \sin^2(x)$ i.e., "Let u = 'the function'"

2. Compute du

$$
\begin{array}{rrcl}
 & u & = & \cos^2(x) - \sin^2(x) = (\cos(x))^2 - (\sin(x))^2 \\
\Rightarrow & \frac{du}{dx} & = & -2\cos(x)\sin(x) - 2\sin(x)\cos(x) = -4\sin(x)\cos(x) \\
\Rightarrow & du & = & -4\sin(x)\cos(x)\,dx \\
\Rightarrow & -\frac{1}{4}du & = & \sin(x)\cos(x)\,dx
\end{array}
$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{\cos^2(x) - \sin^2(x)}}_{\frac{1}{u}} \underbrace{\sin(x)\cos(x)\,dx}_{-\frac{1}{4}du} = \int \frac{1}{u}\left(-\frac{1}{4}du\right) = -\frac{1}{4}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$-\frac{1}{4}\int \frac{1}{u}\,du = -\frac{1}{4}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}\,dx = \underbrace{-\frac{1}{4}\ln\left|\cos^2(x) - \sin^2(x)\right| + C}_{-\frac{1}{4}\ln|u|+C}$$

i.e., $\int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}\,dx = -\frac{1}{4}\ln\left|\cos^2(x) - \sin^2(x)\right| + C$

Alternative Solution Appears on the Next Page

Alternativeley:

I gave this as an extra credit exercise years ago. I decided to give it again, but this time, I didn't see it the way that I had seen it years ago, and the way that some of you saw it on the test. So, for better or for worse, here's how I approached it this time.

Recall: $\sin(2x) = 2\sin(x)\cos(x)$

and: $\cos(2x) = \cos^2(x) - \sin^2(x)$

Thus, $\int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}dx = \int \frac{\frac{1}{2}\sin(2x)}{\cos(2x)}dx = \int \frac{1}{\cos(2x)}\frac{1}{2}\sin(2x)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes. $\frac{1}{\cos(2x)} = (\cos(2x))^{-1}$

      Let $u = \cos(2x)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

      Yes. $\underbrace{\cos(2x)}_{\text{function}} \rightarrow \underbrace{\sin(2x)}_{\text{deriv}}$

      Let $u = \cos(2x)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{lrcl} & u & = & \cos(2x) \\ \Rightarrow & \frac{du}{dx} & = & -2\sin(2x) \\ \Rightarrow & du & = & -2\sin(2x)\,dx \\ \Rightarrow & -\frac{1}{4}du & = & \frac{1}{2}\sin(2x)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{\cos(2x)}}_{\frac{1}{u}}\underbrace{\frac{1}{2}\sin(2x)\,dx}_{-\frac{1}{4}du} = \int \frac{1}{u}\left(-\frac{1}{4}du\right) = -\frac{1}{4}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$-\frac{1}{4}\int \frac{1}{u}\,du = -\frac{1}{4}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}\,dx = \underbrace{-\frac{1}{4}\ln|\cos(2x)| + C}_{-\frac{1}{4}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \frac{\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}\,dx = -\frac{1}{4}\ln|\cos(2x)| + C}$$