MTH 4436 Homework Set 3.3; page 58

Spring 2012

Pat Rossi Name ______________________

1. Verify that the integers 1949 and 1951 are twin primes.

   Observe: $\sqrt{1951} = 44.17$

   To show that 1949 and 1951 are twin primes, we'll show that neither has a prime factor less than $\sqrt{1951} = 44.17$.

p		1949 =		1951 =
2		$1949 = 2(974) + 1$		$1951 = 2(975) + 1$
3		$1949 = 3(649) + 2$		$1951 = 3(650) + 1$
5		$1949 = 5(389) + 4$		$1951 = 5(390) + 1$
7		$1949 = 7(278) + 3$		$1951 = 7(278) + 5$
11		$1949 = 11(177) + 2$		$1951 = 11(177) + 4$
13		$1949 = 13(149) + 12$		$1951 = 13(150) + 1$
17		$1949 = 17(114) + 11$		$1951 = 17(114) + 13$
19		$1949 = 19(102) + 11$		$1951 = 19(102) + 13$
23		$1949 = 23(84) + 17$		$1951 = 23(84) + 19$
29		$1949 = 29(67) + 6$		$1951 = 29(67) + 8$
31		$1949 = 31(62) + 27$		$1951 = 31(62) + 29$
37		$1949 = 37(52) + 25$		$1951 = 37(52) + 27$
41		$1949 = 41(47) + 22$		$1951 = 41(47) + 24$
43		$1949 = 43(45) + 14$		$1951 = 43(45) + 16$

   Since neither 1949 nor 1951 has a prime factor less than $\sqrt{1951} = 44.17$, both are prime. (i.e., 1949 and 1951 are twin primes.)

2. ~

(a) Prove: If 1 is added to the product of twin primes, prove that the result is a perfect square.

Proof. Given any twin prime pair, we can represent the pair as $\langle p, p+2 \rangle$. If we add 1 to the product, we have $p(p+2)+1 = p^2+2p+1 = (p+1)^2$, which is a perfect square. ■

(b) Show that the sum of twin primes, p and $p+2$ is divisible by 12, provided that $p > 3$.

Proof. Let $c = p + (p+2)$, where p and $p+2$ are prime. Since $\gcd(3,4) = 1$, it will follow that if $3|c$ and $4|c$, then $(3 \cdot 4)|c$.
i.e., $12|c$.

Hence our proof boils down to showing that $3|c$ and $4|c$. (i.e., $3|[p+(p+2)]$ and $4|[p+(p+2)]$)

$3|[p+(p+2)]$

Note that p must have one of the following three forms:

$p = 3k$; $p = 3k+1$; $p = 3k+2$, for some natural number k.

Case 1: $p = 3k$

This can't happen, because this would make p composite, contrary to hypothesis.

Case 2: $p = 3k+1$

This can't happen, because this would make $p+2 = (3k+1)+3 = 3k+3 = 3(k+1)$.

i.e., $p+2$ would be composite, contrary to hypothesis.

This leaves ...

Case 3: $p = 3k+2$ (This MUST be the case!)

Hence, $p+2 = (3k+2)+2 = 3k+4$.

Observe: $p+(p+2) = (3k+2)+(3k+4) = 3k+6 = 3(k+2)$

Thus, $3|[p+(p+2)]$

$4|[p+(p+2)]$

Since $p > 3$, p must be odd. Hence, $p = 2k+1$, for some natural number k.

Thus, $p+(p+2) = (2k+1)+[(2k+1)+2] = 4k+4 = 4(k+1)$

Thus, $4|[p+(p+2)]$, and our claim is proved. ■

3. Find all pairs of primes p and q satisfying $p - q = 3$.

Observe: The difference of two even numbers is an even number. The difference of two odd numbers is an even number. Since $p - q$ is odd, one of these must be even, and the other odd. Since 2 is the only even prime, and since 2 is the smallest prime, we must have $q = 2$. Hence, $p = 5$.

This is the only such pair of primes p and q satisfying $p - q = 3$.

9. (a) For $n > 3$, show that $n, n+2$, and $n+4$ cannot all be prime.

Proof. Let $n > 3$.

Note that n must have on of the following three forms:

$n = 3k$; $\quad n = 3k+1$; $\quad n = 3k+2$, $\quad$ for some natural number k.

Case 1, $n = 3k$

In this case, n is not prime, as $3|n$

Case 2, $n = 3k+1$

Then $n + 2 = (3k+1) + 2 = 3k + 3 = 3(k+1)$.

i.e., $n + 2 = 3(k+1)$.

In this case, $n+2$ is not prime, as $3|(n+2)$

Case 3, $n = 3k+2$

Then $n + 4 = (3k+2) + 4 = 3k + 6 = 3(k+2)$

i.e., $n + 4 = 3(k+2)$

In this case, $n+4$ is not prime, as $3|(n+4)$ ■

b. Three integers $p, p+2, p+6$, which are all prime, are called a *prime triplet.* Find five sets of prime triplets.

Observe: since $p, p+2$ constitute a twin prime pair, consider all twin prime pairs.

p		$p+2$		$p+6$
3		5		9 No!
5		7		11 Yes!
11		13		17 Yes!
17		19		23 Yes!
29		31		35 No!
41		43		47 Yes!
59		61		65 No!
71		73		77 No!
101		103		107 Yes!