Taylor Series Exercises - Solutions

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In Exercises 1-3, give the well-known Taylor Series expansion (with center $c = 0$) for $f(x)$.

1. $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\boxed{\text{i.e., } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots}$$

2. $f(x) = \sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = (-1)^0 \frac{x^{(2(0)+1)}}{(2(0)+1)!} + (-1)^1 \frac{x^{(2(2)+1)}}{(2(2)+1)!} + (-1)^2 \frac{x^{(2(2)+1)}}{(2(2)+1)!}$$

$$+ (-1)^3 \frac{x^{(2(3)+1)}}{(2(3)+1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$\boxed{\text{i.e., } \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots}$$

3. $f(x) = \cos(x)$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = (-1)^0 \frac{x^{2(0)}}{(2(0))!} + (-1)^1 \frac{x^{2(2)}}{(2(2))!} + (-1)^2 \frac{x^{2(3)}}{2(3)!}$$

$$+ (-1)^3 \frac{x^{2(3)}}{(2(3))!} + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\boxed{\text{i.e., } \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots}$$

In Exercises 4-8, use a well-known Taylor Series representation to derive a Taylor Series representation for the function given.

4. e^{5x}

   To derive a Taylor Series representation for e^{5x}, we will use the known Taylor Series representation for the function $f(x) = e^x$.

   Observe:

   $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

   We want: e^{5x}

   But e^{5x} is just $f(5x) = e^{5x}$

   i.e., e^{5x} is $f(x) = e^x$, with $5x$ "playing the role" of x.

   Therefore: $e^{5x} = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \ldots + \frac{(5x)^n}{n!} + \ldots$

   $$\boxed{e^{5x} = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \ldots + \frac{(5x)^n}{n!} + \ldots}$$

5. $\sin(3x^2)$

   To derive a Taylor Series representation for $\sin(3x^2)$, we will use the known Taylor Series representation for the function $f(x) = \sin(x)$.

   Observe:

   $f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$

   We want: $\sin(3x^2)$

   But $\sin(3x^2)$ is just $f(3x^2) = \sin(3x^2)$

   i.e., $\sin(3x^2)$ is $f(x) = \sin(x)$, with $3x^2$ "playing the role" of x.

   Therefore: $\sin(3x^2) = 3x^2 - \frac{(3x^2)^3}{3!} + \frac{(3x^2)^5}{5!} - \frac{(3x^2)^7}{7!} + \ldots + (-1)^n \frac{(3x^2)^{2n+1}}{(2n+1)!} + \ldots$

   $$\boxed{\sin(3x^2) = 3x^2 - \frac{(3x^2)^3}{3!} + \frac{(3x^2)^5}{5!} - \frac{(3x^2)^7}{7!} + \ldots + (-1)^n \frac{(3x^2)^{2n+1}}{(2n+1)!} + \ldots}$$

6. $\cos(2x-5)$

To derive a Taylor Series representation for $\cos(2x-5)$, we will use the known Taylor Series representation for the function $f(x) = \cos(x)$.

Observe:

$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$

We want: $\cos(2x-5)$

But $\cos(2x-5)$ is just $f(2x-5) = \cos(2x-5)$

i.e., $\cos(2x-5)$ is $f(x) = \cos(x)$, with $2x-5$ "playing the role" of x.

Therefore: $\cos(2x-5) = 1 - \frac{(2x-5)^2}{2!} + \frac{(2x-5)^4}{4!} - \frac{(2x-5)^6}{6!} + \ldots + (-1)^n \frac{(2x-5)^{2n}}{(2n)!} + \ldots$

$$\boxed{\cos(2x-5) = 1 - \frac{(2x-5)^2}{2!} + \frac{(2x-5)^4}{4!} - \frac{(2x-5)^6}{6!} + \ldots + (-1)^n \frac{(2x-5)^{2n}}{(2n)!} + \ldots}$$

7. x^2e^x

To derive a Taylor Series representation for x^2e^x, we will use the known Taylor Series representation for the function $f(x) = e^x$.

Observe:

$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

We want: x^2e^x

But x^2e^x is just $x^2 \cdot f(x) = x^2 \cdot e^x$

Therefore: $x^2e^x = x^2 \cdot \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right)$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \ldots + \frac{x^{n+2}}{n!} + \ldots$$

$$\boxed{x^2e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \ldots + \frac{x^{n+2}}{n!} + \ldots}$$

8. $\frac{1}{x}\sin(x)$

To derive a Taylor Series representation for $\frac{1}{x}\sin(x)$, we will use the known Taylor Series representation for the function $f(x) = \sin(x)$.

Observe:

$f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$

We want: $\frac{1}{x}\sin(x)$

But $\frac{1}{x}\sin(x)$ is just $\frac{1}{x} \cdot f(x) = \frac{1}{x} \cdot \sin(x)$

Therefore: $\frac{1}{x}\sin(x) = \frac{1}{x} \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\right)$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \ldots$$

$$\boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \ldots}$$