MTH 1125 Test #1

Summer 2010 - Solutions

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2} \frac{x^2+x+4}{x^3-3x+4} =$

   1. Try Plugging in:

      $\lim_{x\to 2} \frac{x^2+x+4}{x^3-3x+4} = \frac{(2)^2+(2)+4}{(2)^3-3(2)+4} = \frac{5}{3}$

      i.e., $\lim_{x\to 2} \frac{x^2+x+4}{x^3-3x+4} = \frac{5}{3}$

2. Compute: $\lim_{x\to -3} \frac{x^2-x-12}{x^2+x-6} =$

   1. Try Plugging in:

      $\lim_{x\to -3} \frac{x^2-x-12}{x^2+x-6} = \frac{(-3)^2-(-3)-12}{(-3)^2+(-3)-6} = \frac{0}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

      $\lim_{x\to -3} \frac{x^2-x-12}{x^2+x-6} = \lim_{x\to -3} \frac{(x+3)(x-4)}{(x+3)(x-2)} = \lim_{x\to -3} \frac{(x-4)}{(x-2)} = \frac{(-3)-4}{(-3)-2} = \frac{7}{5}$

      i.e., $\lim_{x\to -3} \frac{x^2-x-12}{x^2+x-6} = \frac{7}{5}$

3. Compute: $\lim_{x\to -\infty} \frac{5x^3+3x^2+5}{5x^4-7x^3+5x^2+x} =$

   $\lim_{x\to -\infty} \frac{5x^3+3x^2+5}{5x^4-7x^3+5x^2+x} = \lim_{x\to -\infty} \frac{5x^3}{5x^4} = \lim_{x\to -\infty} \frac{1}{x} = 0$

   i.e., $\lim_{x\to -\infty} \frac{5x^3+3x^2+5}{5x^4-7x^3+5x^2+x} = 0$

4. Compute: $\lim_{x \to 2} \frac{x+5}{x^2+x-6} =$

   1. Try Plugging in:

      $\lim_{x \to 2} \frac{x+5}{x^2+x-6} = \frac{(2)+5}{(2)^2+(2)-6} = \frac{7}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

      No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

   3. Analyze the one-sided limits:

      $\lim_{x \to 2^-} \frac{x+5}{x^2+x-6} = \lim_{x \to 2^-} \frac{x+5}{(x+3)(x-2)} = \frac{7}{(5)(-\varepsilon)} = \frac{\left(\frac{7}{5}\right)}{(-\varepsilon)} = -\infty$

      $$\begin{array}{|ll|} & x \longrightarrow 2^- \\ \Rightarrow & x < 2 \\ \Rightarrow & x - 2 < 0 \end{array}$$

      $\lim_{x \to 2^+} \frac{x+5}{x^2-x-6} = \lim_{x \to 2^+} \frac{x+5}{(x+3)(x-2)} = \frac{7}{(5)(+\varepsilon)} = \frac{\left(\frac{7}{5}\right)}{(+\varepsilon)} = +\infty$

      $$\begin{array}{|ll|} & x \longrightarrow 2^+ \\ \Rightarrow & x > 2 \\ \Rightarrow & x - 2 > 0 \end{array}$$

Since the one-sided limits are not equal, $\lim_{x \to 2} \frac{x+5}{x^2+x-6}$ **Does Not Exist!**

5. $f(x) = 6x^4 + 8x^3 + 12x^2 + 24x + 5;$ Compute: $f'(x)$.

   $f'(x) = 6(4x^3) + 8(3x^2) + 12(2x^1) + 24(1) + 0 = 24x^3 + 24x^2 + 24x + 24$

   i.e., $f'(x) = 24x^3 + 24x^2 + 24x + 24$

6. $\frac{d}{dx}[8\sin(x) - 5\cos(x)] =$

   $\frac{d}{dx}[8\sin(x) - 5\cos(x)] = 8(\cos(x)) - 5(-\sin(x)) = 8\cos(x) + 5\sin(x)$

   i.e., $\frac{d}{dx}[8\sin(x) - 5\cos(x)] = 8\cos(x) + 5\sin(x)$

7. Find the asymptotes and graph: $f(x) = \frac{x-1}{x^2-x-6}$

Verticals

1. Find x-values that cause division by zero.

$\Rightarrow x^2 - x - 6 =$

$\Rightarrow (x+2)(x-3) = 0$

$\Rightarrow x = -2$ and $x = 3$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$$\lim_{x\to -2^-} \frac{x-1}{x^2-x-6} = \lim_{x\to -2^-} \frac{x-1}{(x+2)(x-3)} = \frac{-3}{(-\varepsilon)(-5)} = \frac{\left(\frac{-3}{-5}\right)}{(-\varepsilon)} = \frac{\left(\frac{3}{5}\right)}{-\varepsilon} = -\infty$$

$$\begin{array}{|ll|} & x \to -2^- \\ \Rightarrow & x < -2 \\ \Rightarrow & x+2 < 0 \end{array}$$

$$\lim_{x\to -2^+} \frac{x-1}{x^2-x-6} = \lim_{x\to -2^+} \frac{x-1}{(x+2)(x-3)} = \frac{-3}{(\varepsilon)(-5)} = \frac{3}{(\varepsilon)(5)} = \frac{\left(\frac{3}{5}\right)}{\varepsilon} = +\infty$$

$$\begin{array}{|ll|} & x \to -2^+ \\ \Rightarrow & x > -2 \\ \Rightarrow & x+2 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = -2$ *IS a vertical asymptote.*

$$\lim_{x\to 3^-} \frac{x-1}{x^2-x-6} = \lim_{x\to 3^-} \frac{x-1}{(x+2)(x-3)} = \frac{2}{(5)(-\varepsilon)} = \frac{\left(\frac{2}{5}\right)}{(-\varepsilon)} = -\infty$$

$$\begin{array}{|ll|} & x \to 3^- \\ \Rightarrow & x < 3 \\ \Rightarrow & x-3 < 0 \end{array}$$

$$\lim_{x\to 3^+} \frac{x-1}{x^2-x-6} = \lim_{x\to 3^+} \frac{x-1}{(x+2)(x-3)} = \frac{2}{(5)(+\varepsilon)} = \frac{\left(\frac{2}{5}\right)}{(+\varepsilon)} = +\infty$$

$$\begin{array}{|ll|} & x \to 3^+ \\ \Rightarrow & x > 3 \\ \Rightarrow & x-3 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = 3$ *IS a vertical asymptote.*

Horizontals

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{x-1}{x^2-x-6} = \lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0$

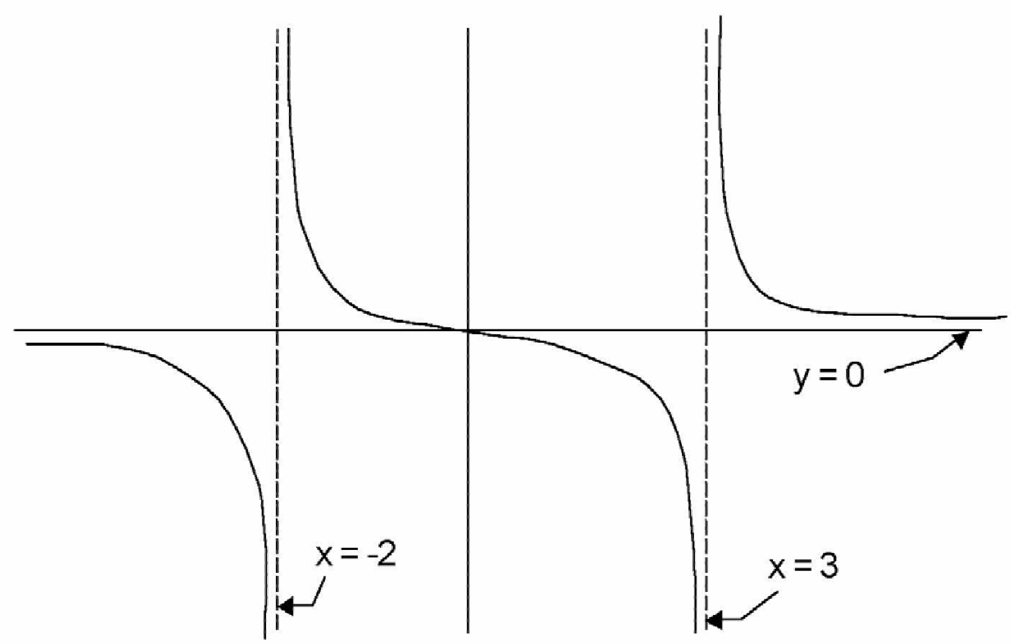
$\lim_{x\to+\infty} \frac{x-1}{x^2-x-6} = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0$

Since the limits are finite and constant, $y = 0$ is a horizontal asymptote.

Summary:

$\lim_{x\to-2^-} \frac{x-1}{x^2-x-6} = -\infty$	
$\lim_{x\to-2^+} \frac{x-1}{x^2-x-6} = +\infty$	$\lim_{x\to-\infty} \frac{x-1}{x^2-x-6} = 0$
$\lim_{x\to3^-} \frac{x-1}{x^2-x-6} = -\infty$	$\lim_{x\to+\infty} \frac{x-1}{x^2-x-6} = 0$
$\lim_{x\to3^+} \frac{x-1}{x^2-x-6} = +\infty$	

Graph $f(x) = \frac{x-1}{x^2-x-6}$



8.

$x =$	$f(x) =$
3.5	−15.1
3.9	−227.8
3.99	−1212.3
3.999	−21156.3
3.9999	−834561.9

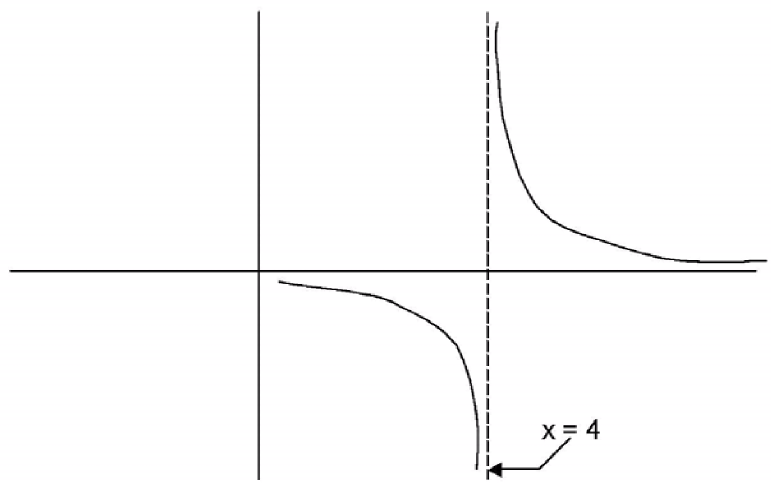
$x =$	$f(x) =$
4.5	15.1
4.1	227.8
4.01	1212.3
4.001	21156.3
4.0001	834561.9

Based on the information in the table above, do the following:

(a) $\lim_{x \to 4^-} f(x) = -\infty$

(b) $\lim_{x \to 4^+} f(x) = \infty$

(c) Graph $f(x)$



9. Determine whether or not $f(x)$ is continuous at the point $x = 4$. (Justify your answer.)

$$f(x) = \begin{cases} \frac{x^2-x-12}{x-4} & \text{for } x \leq 4 \\ 8 - x & \text{for } x > 4 \end{cases}$$

If $f(x)$ is continuous at the point $x = 4$, then $\lim_{x\to 4} f(x) = f(4)$.

To see if this is true, we'll compute $\lim_{x\to 4} f(x)$.

Since the definiton of $f(x)$ changes at $x = 4$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x^2-x-12}{x-4} = \lim_{x\to 4^-} \frac{(x+3)(x-4)}{x-4} = \lim_{x\to 4^-} (x+3) = 7$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (8 - x) = 4$

Since the one-sided limits are NOT equal, $\lim_{x\to 4} f(x)$ DOES NOT EXIST.

Therefore, $\lim_{x\to 4} f(x) \neq f(4)$

Hence, $f(x)$ is NOT continuous at $x = 4$

10. $f(x) = 2x^2 + 6x + 3$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[2(x+\Delta x)^2+6(x+\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[2\left(x^2+2x\Delta x+\Delta x^2\right)+6(x+\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[\left(2x^2+4x\Delta x+2\Delta x^2\right)+(6x+6\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{4x\Delta x+2\Delta x^2+6\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(4x+2\Delta x+6)}{\Delta x} = \lim_{\Delta x\to 0} (4x + 2\Delta x + 6) = 4x + 2(0) + 6 = 4x + 6$

$\boxed{\text{i.e., } f'(x) = 4x + 6}$

11. Compute: $\lim_{x\to 5}\frac{\sqrt{20+x}-5}{x-5} =$

1. Try Plugging in:

   $\lim_{x\to 5}\frac{\sqrt{20+x}-5}{x-5} = \frac{\sqrt{20+(5)}-5}{(5)-5} = \frac{0}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

   $$\lim_{x\to 5}\frac{\sqrt{20+x}-5}{x-5} = \lim_{x\to 5}\frac{\sqrt{20+x}-5}{x-5}\cdot\frac{\sqrt{20+x}+5}{\sqrt{20+x}+5} = \lim_{x\to 5}\frac{\left(\sqrt{20+x}\right)^2-(5)^2}{(x-5)\left[\sqrt{20+x}+5\right]}$$

   $$= \lim_{x\to 5}\frac{20+x-25}{(x-5)\left[\sqrt{20+x}+5\right]} = \lim_{x\to 5}\frac{(x-5)}{(x-5)\left[\sqrt{20+x}+5\right]} = \lim_{x\to 5}\frac{1}{\sqrt{20+x}+5}$$

   $$= \frac{1}{\sqrt{20+(5)}+5} = \frac{1}{5+5} = \frac{1}{10}$$

   $$\boxed{\text{i.e., } \lim_{x\to 5}\frac{\sqrt{20+x}-5}{x-5} = \frac{1}{10}}$$

EXTRA - WOW! (7 pts) Compute: $\lim_{x\to\infty}\frac{\sin(x)}{x} =$ (Justify your answer completely)

Observe: $-1 \le \sin(x) \le 1$

$\Rightarrow\quad -\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$$\Rightarrow\quad \underbrace{\lim_{x\to\infty} -\frac{1}{x}}_{=0} \le \lim_{x\to\infty}\frac{\sin(x)}{x} \le \underbrace{\lim_{x\to\infty}\frac{1}{x}}_{=0}$$

i.e., $0 \le \lim_{x\to\infty}\frac{\sin(x)}{x} \le 0$

$\Rightarrow\quad \lim_{x\to\infty}\frac{\sin(x)}{x} = 0$