Solutions to Selected Exercises on p. 28

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3a Express the fraction $\frac{1}{6}$ in sexagesimal notation

Our aim is to express $\frac{1}{6}$ as the sum of powers of 60

We want x such that $\frac{1}{6} = \frac{x}{60}$

Therefore, we must express $\frac{1}{6}$ as a fraction with a denominator of 60.

Observe: $\frac{1}{6} = \frac{1}{6} \cdot \underbrace{\frac{10}{10}}_{=1} = \frac{10}{60} = 0;10$

$$\boxed{\text{i.e., } \frac{1}{6} = 0;10}$$

(Ex 3b is on the next page)

3b Express the fraction $\frac{1}{9}$ in sexagesimal notation

Our aim is to express $\frac{1}{9}$ as the sum of powers of 60

We want x such that $\frac{1}{9} = \frac{x}{60}$; or $\frac{1}{9} = \frac{x}{60} + \frac{y}{60^2}$; etc.

Observe: $\frac{1}{9} = \frac{1}{3^2}$ is a fraction in "reduced form" (numerator and denominator have no prime factors in common).

Since 60 only has **one** factor of 3 ($60 = 2^2 \cdot 3 \cdot 5$) and 9 has two factors of 3 ($9 = 3^2$) , the (reduced) fraction $\frac{1}{9}$ cannot be expressed solely as a single fraction with denominator 60.

i.e. There is no whole number x such that $\frac{1}{9} = \frac{x}{60}$.

Therefore, we must express $\frac{1}{9}$ as the sum of two or more fractions whose denominators are powers of 60.

If we factor the denominators 9 and 60, and compare, here's what we see:

$9 = 3^2$; and $60 = 2^2 \cdot 3 \cdot 5$

9 has a factor of 3 that 60 doesn't have and 60 has factors 2^2 and 5 that 9 doesn't have.

Therefore, in order to express $\frac{1}{9}$ as fraction whose denominator is a power of 60, that power of 60 must contain all factors of 9.

Observe that $60^2 = \left(2^2 \cdot 3 \cdot 5\right)^2 = 2^4 \cdot 3^2 \cdot 5^2$ contains all factors of 9.

Therefore, we can express $\frac{1}{9}$ as a fraction with denominator of 60^2 by multiplying numerator and denominator of $\frac{1}{9}$ by those factors of 60^2 that are not factors of 9.

i.e., $\frac{1}{9} = \frac{1}{9} \cdot \frac{2^4 \cdot 5^2}{2^4 \cdot 5^2} = \frac{2^4 \cdot 5^2}{9 \cdot 2^4 \cdot 5^2} = \frac{400}{3600} = \frac{400}{60^2}$

i.e., $\frac{1}{9} = \frac{400}{60^2}$

We should now express the numerator as the sum of powers of 60, each with a coefficient less than or equal to 60 (because we will be expressing the number in base 60).

i.e., $\frac{1}{9} = \frac{400}{60^2} = \frac{6 \cdot 60 + 40}{60^2} = \frac{6 \cdot 60}{60^2} + \frac{40}{60^2} = \frac{6}{60} + \frac{40}{60^2} = 0; 6, 40$

i.e., $\frac{1}{9} = 0; 6, 40$

3c Express the fraction $\frac{1}{5}$ in sexagesimal notation

Our aim is to express $\frac{1}{5}$ as the sum of powers of 60

We want x such that $\frac{1}{5} = \frac{x}{60}$

Therefore, we must express $\frac{1}{5}$ as a fraction with a denominator of 60.

Observe: $\frac{1}{5} = \frac{1}{5} \cdot \underbrace{\frac{12}{12}}_{=1} = \frac{12}{60} = 0;12$

i.e., $\frac{1}{5} = 0;12$

(Ex 3d is on the next page)

3d Express the fraction $\frac{1}{24}$ in sexagesimal notation

Our aim is to express $\frac{1}{24}$ as the sum of powers of 60

We want x such that $\frac{1}{24} = \frac{x}{60}$; or $\frac{1}{24} = \frac{x}{60} + \frac{y}{60^2}$; etc.

Observe: $\frac{1}{24} = \frac{1}{2^3 \cdot 3}$ is a fraction in "reduced form" (numerator and denominator have no prime factors in common).

Since 60 only has **two** factors of 2 $(60 = 2^2 \cdot 3 \cdot 5)$ and 24 has **three** factors of 2 $(24 = 2^3 \cdot 3)$, the (reduced) fraction $\frac{1}{24}$ cannot be expressed solely as a single fraction with denominator 60.

i.e. There is no whole number x such that $\frac{1}{24} = \frac{x}{60}$.

Therefore, we must express $\frac{1}{24}$ as the sum of two or more fractions whose denominators are powers of 60.

If we factor the denominators 24 and 60, and compare, here's what we see:

$24 = 2^3 \cdot 3$; and $60 = 2^2 \cdot 3 \cdot 5$

24 has a factor of 2 that 60 doesn't have and 60 has a factor of 5 that 24 doesn't have.

Therefore, in order to express $\frac{1}{24}$ as fraction whose denominator is a power of 60, that power of 60 must contain all factors of 24.

Observe that $60^2 = \left(2^2 \cdot 3 \cdot 5\right)^2 = 2^4 \cdot 3^2 \cdot 5^2$ contains all factors of 24.

Therefore, we can express $\frac{1}{24}$ as a fraction with denominator of 60^2 by multiplying numerator and denominator of $\frac{1}{24}$ by those factors of 60^2 that are not factors of 24.

i.e., $\frac{1}{24} = \frac{1}{24} \cdot \frac{2 \cdot 3 \cdot 5^2}{2 \cdot 3 \cdot 5^2} = \frac{2 \cdot 3 \cdot 5^2}{24 \cdot 2 \cdot 3 \cdot 5^2} = \frac{150}{3600} = \frac{150}{60^2}$

i.e., $\frac{1}{24} = \frac{150}{60^2}$

We should now express the numerator as the sum of powers of 60, each with a coefficient less than or equal to 60 (because we will be expressing the number in base 60).

i.e., $\frac{1}{24} = \frac{150}{60^2} = \frac{2 \cdot 60 + 30}{60^2} = \frac{2 \cdot 60}{60^2} + \frac{30}{60^2} = \frac{2}{60} + \frac{30}{60^2} = 0; 2, 30$

i.e., $\frac{1}{24} = 0; 2, 30$

3e Express the fraction $\frac{1}{40}$ in sexagesimal notation

Our aim is to express $\frac{1}{40}$ as the sum of powers of 60

We want x such that $\frac{1}{40} = \frac{x}{60}$; or $\frac{1}{40} = \frac{x}{60} + \frac{y}{60^2}$; etc.

Observe: $\frac{1}{40} = \frac{1}{2^3 \cdot 5}$ is a fraction in "reduced form" (numerator and denominator have no prime factors in common).

Since 60 only has **two** factors of 2 $(60 = 2^2 \cdot 3 \cdot 5)$ and 40 has **three** factors of 2 $(40 = 2^3 \cdot 5)$, the (reduced) fraction $\frac{1}{40}$ cannot be expressed solely as a single fraction with denominator 60.

i.e. There is no whole number x such that $\frac{1}{40} = \frac{x}{60}$.

Therefore, we must express $\frac{1}{40}$ as the sum of two or more fractions whose denominators are powers of 60.

If we factor the denominators 40 and 60, and compare, here's what we see:

$40 = 2^3 \cdot 5$; and $60 = 2^2 \cdot 3 \cdot 5$

40 has a factor of 2 that 60 doesn't have and 60 has a factor of 3 that 40 doesn't have.

Therefore, in order to express $\frac{1}{40}$ as fraction whose denominator is a power of 60, that power of 60 must contain all factors of 40.

Observe that $60^2 = \left(2^2 \cdot 3 \cdot 5\right)^2 = 2^4 \cdot 3^2 \cdot 5^2$ contains all factors of 40.

Therefore, we can express $\frac{1}{40}$ as a fraction with denominator of 60^2 by multiplying numerator and denominator of $\frac{1}{40}$ by those factors of 60^2 that are not factors of 40.

i.e., $\frac{1}{40} = \frac{1}{40} \cdot \frac{2 \cdot 3^2 \cdot 5}{2 \cdot 3^2 \cdot 5} = \frac{2 \cdot 3^2 \cdot 5}{40 \cdot 2 \cdot 3^2 \cdot 5} = \frac{90}{3600} = \frac{90}{60^2}$

i.e., $\frac{1}{40} = \frac{90}{60^2}$

We should now express the numerator as the sum of powers of 60, each with a coefficient less than or equal to 60 (because we will be expressing the number in base 60).

i.e., $\frac{1}{40} = \frac{90}{60^2} = \frac{1 \cdot 60 + 30}{60^2} = \frac{1 \cdot 60}{60^2} + \frac{30}{60^2} = \frac{1}{60} + \frac{30}{60^2} = 0; 1, 30$

i.e., $\frac{1}{40} = 0; 1, 30$

3f Express the fraction $\frac{5}{12}$ in sexagesimal notation

Our aim is to express $\frac{5}{12}$ as the sum of powers of 60

We want x such that $\frac{5}{12} = \frac{x}{60}$

Therefore, we must express $\frac{5}{12}$ as a fraction with a denominator of 60.

Observe: $\frac{5}{12} = \frac{5}{12} \cdot \underbrace{\frac{5}{5}}_{=1} = \frac{25}{60} = 0; 25$

i.e., $\frac{5}{12} = 0; 25$

(Ex 4a is on the next page)

4a Convert the number $1, 23, 45.$ from sexagesimal to "our system" (i.e., to decimal (base 10)).

$$1, 23, 45. = 1 \cdot 60^2 + 23 \cdot 60 + 45 = 5, 025$$

i.e., $1, 23, 45. = 5, 025$

(Ex 4b is on the next page)

4b Convert the number $12;3,45$ from sexagesimal to "our system" (i.e., to decimal (base 10)).

$12;3,45 = 12 \cdot 60^0 + 3 \cdot \frac{1}{60} + 45 \cdot \frac{1}{60^2} = 12 + \frac{3}{60} + \frac{45}{60^2}$

$= 12 + \frac{3}{3 \cdot 20} + \frac{45}{(3 \cdot 20)^2}$

Observe: Because we want to convert to base 10, we would like for the fractions $\frac{3}{2^2 \cdot 5}$ and $\frac{45}{2^4 \cdot 5^2}$ to be expressed as fractions whose denominators are powers of 10.

Taking into consideration that the prime factorization of 10 is $10 = 2 \cdot 5$, we have:

$12;3,45 = 12 + \frac{3}{3 \cdot 20} + \frac{45}{(3 \cdot 20)^2} = 12 + \frac{3}{3 \cdot 2^2 \cdot 5} + \frac{45}{(3 \cdot 2^2 \cdot 5)^2} = 12 + \frac{3}{3 \cdot 2^2 \cdot 5} + \frac{3^2 \cdot 5}{3^2 \cdot 2^4 \cdot 5^2}$

$= 12 + \frac{1}{2^2 \cdot 5} + \frac{5}{2^4 \cdot 5^2} = 12 + \frac{1}{2^2 \cdot 5} \cdot \frac{5}{5} + \frac{5}{2^4 \cdot 5^2} \cdot \frac{5^2}{5^2} = 12 + \frac{5}{2^2 \cdot 5^2} + \frac{125}{2^4 \cdot 5^4}$

$= 12 + \frac{5}{(2 \cdot 5)^2} + \frac{125}{(2 \cdot 5)^4} = 12 + \frac{5}{10^2} + \frac{125}{10^4} = 12 + \frac{5}{10^2} + \frac{100}{10^4} + \frac{25}{10^4}$

$= 12 + \frac{5}{100} + \frac{1}{100} + \frac{25}{10^4} = 12 + \frac{6}{100} + \frac{25}{10^4} = 12 + \frac{6}{10^2} + \frac{2 \cdot 10 + 5}{10^4}$

$= 12 + \frac{6}{10^2} + \frac{2 \cdot 10}{10^4} + \frac{5}{10^4} = 12 + \frac{6}{10^2} + \frac{2}{10^3} + \frac{5}{10^4} = 12.0625$

Because we want to convert to base 10, we would like for the fractions $\frac{1}{2^2 \cdot 5}$ and $\frac{5}{2^4 \cdot 5^2}$ to be expressed as fractions whose denominators are powers of 10 (i.e. powers of $2 \cdot 5$).

Thus: $12;3,45 = 12.0625$

i.e., $12;3,45 = 12.0625$

Alternatively:

$12;3,45 = 12 + \frac{3}{60} + \frac{45}{60^2} = 12 + \frac{1}{20} + \frac{5}{20 \cdot 20} = 12 + \frac{1}{20} + \frac{1}{80}$

$= 12 + \frac{4}{80} + \frac{1}{80} = 12 + \frac{5}{80} = 12 + \frac{1}{16}$

i.e., $12;3,45 = 12 + \frac{1}{16}$

Remark: Since the direction given stated: "Convert these numbers form sexagesimal notation to 'our system' (i.e. base 10), the former answer ($12;3,45 = 12.0625$) is preferable.

4c Convert the number 0; 12, 3, 45 from sexagesimal to "our system" (i.e., to decimal (base 10)).

Observe: $0;12,3,45 = \frac{12}{60} + \frac{3}{60^2} + \frac{45}{60^3}$

(Eliminate, from the denominator, all prime factors that are not factors of 10)

$0;12,3,45 = \frac{12}{60} + \frac{3}{60^2} + \frac{45}{60^3} = \frac{2^2 \cdot 3}{2^2 \cdot 3 \cdot 5} + \frac{3}{(2^2 \cdot 3 \cdot 5)^2} + \frac{3^2 \cdot 5}{(2^2 \cdot 3 \cdot 5)^3}$

$= \frac{2^2 \cdot 3}{2^2 \cdot 3 \cdot 5} + \frac{3}{2^4 \cdot 3^2 \cdot 5^2} + \frac{3^2 \cdot 5}{2^6 \cdot 3^3 \cdot 5^3} = \frac{2^2 \cdot 3}{2^2 \cdot 3 \cdot 5} + \frac{3}{2^4 \cdot 3^2 \cdot 5^2} + \frac{3^2 \cdot 5}{2^6 \cdot 3^3 \cdot 5^3}$

$= \frac{2^2}{2^2 \cdot 5} + \frac{1}{2^4 \cdot 3 \cdot 5^2} + \frac{5}{2^6 \cdot 3 \cdot 5^3} = \frac{2}{2 \cdot 5} + \frac{1}{2^4 \cdot 3 \cdot 5^2} + \frac{1}{2^6 \cdot 3 \cdot 5^2}$

OOPS! These fractions are "reduced to lowest terms" and 3 is a factor of two of the denominators. Since 3 is a prime factor that is not a factor of 10, we will not be able to express these fractions in such a way that their denominators are powers of 10.

i.e., We won't be able to express 0; 12, 3, 45 in "decimal form" in base 10. The best that we'll be able to do is express 0; 12, 3, 45 as a whole number in base10 plus a fraction whose denominator is not a power of 10.

Consequently, we proceed as follows:

$0;12,3,45 = \frac{12}{60} + \frac{3}{60^2} + \frac{45}{60^3} = \frac{12}{60}\frac{60^2}{60^2} + \frac{3}{60^2}\frac{60}{60} + \frac{45}{60^3} = \frac{12 \cdot 60^2 + 3 \cdot 60 + 45}{60^3}$

$= \frac{43425}{60^3} = \frac{3^2 5^2 193}{2^6 3^3 5^3} = \frac{193}{2^6 \cdot 3 \cdot 5} = \frac{193}{960}$

i.e., $0;12,3,45 = \frac{193}{960}$

4d Convert the number $1,23;45$ from sexagesimal to "our system" (i.e., to decimal (base 10)).

Observe:

$1,23;45 = 1 \cdot 60 + 23 + \frac{45}{60} = 1 \cdot 60 + 23 + \frac{3^2 \cdot 5}{2^2 \cdot 3 \cdot 5} = 1 \cdot 60 + 23 + \frac{3 \cdot 5}{2^2 \cdot 5}$

$= 83 + \frac{3 \cdot 5}{2^2 \cdot 5}$

Observe also: Because we want to convert to base 10, we would like for the fraction $\frac{3 \cdot 5}{2^2 \cdot 5}$ to be expressed as a fraction whose denominator is a power of 10.

Taking into consideration the fact that the prime factorization of 10 is $10 = 2 \cdot 5$, we have:

$1,23;45 = 83 + \frac{3 \cdot 5}{2^2 \cdot 5} = 83 + \frac{3 \cdot 5}{2^2 \cdot 5} \cdot \frac{5}{5} = 83 + \frac{3 \cdot 5^2}{2^2 \cdot 5^2} = 83 + \frac{3 \cdot 5^2}{(2 \cdot 5)^2}$

$= 83 + \frac{75}{10^2} = 83.75$

i.e., $1,23;45 = 83.75$

Alternatively:

$1,23;45 = 1 \cdot 60 + 23 + \frac{45}{60} = 1 \cdot 60 + 23 + \frac{3^2 \cdot 5}{2^2 \cdot 3 \cdot 5} = 1 \cdot 60 + 23 + \frac{3}{2^2}$

$= 83\frac{3}{4}$

i.e., $1,23;45 = 83\frac{3}{4}$

Remark: Since the direction given stated: "Convert these numbers form sexagesimal notation to 'our system' (i.e. base 10), the former answer ($1,23;45 = 83.75$) is preferable.