MTH 3311 Test #2

SUMMER 2016

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Show CLEARLY how you arrive at your answers

1. Solve, using the Method of Undetermined Coefficients:

   $y'' + y' - 6y = 36\cos 3x - 24\sin 3x$

   First, find the solution to the complementary equation $y'' + y' - 6y = 0$

   $\Rightarrow m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m_1 = -3$ and $m_2 = 2$

   $\Rightarrow y_c = c_1 e^{-3x} + c_2 e^{2x}$

   For the particular solution, we imagine that $y_p = A\sin(3x) + B\cos(3x)$

   $y_p' = 3A\cos(3x) - 3B\sin(3x)$

   $y_p'' = -9A\sin(3x) - 9B\cos(3x)$

   To find A and B, we plug these into the original equation, $y'' + y' - 6y = 36\cos 3x - 24\sin 3x$.

   This yields:

$$\underbrace{-9A\sin(3x) - 9B\cos(3x)}_{y''} + \underbrace{3A\cos(3x) - 3B\sin(3x)}_{y'} - \underbrace{6(A\sin(3x) + B\cos(3x))}_{6y} = 36\cos 3x - 24\sin 3x$$

   $\Rightarrow (-9A - 3B - 6A)\sin(3x) + (-9B + 3A - 6B)\cos(3x) = 36\cos 3x - 24\sin 3x$

   $\Rightarrow (-9A - 3B - 6A) = -24$ and $(-9B + 3A - 6B) = 36$

   i.e., $(-15A - 3B) = -24$ (eq. 1) and $(-15B + 3A) = 36$ (eq.2)

   From eq.1, we have: $-3B = -24 + 15A \Rightarrow B = 8 - 5A$

   Plugging this into eq. 2, we have:

   $(-15(8 - 5A) + 3A) = 36 \Rightarrow -120 + 78A = 36 \Rightarrow 78A = 156$

   $\Rightarrow A = 2$

   Plugging this into eq. 2, we have:

   $(-15B + 3(2)) = 36 \Rightarrow -15B = 30$

   $\Rightarrow B = -2$

   Hence, $y_p = 2\sin(3x) - 2\cos(3x)$

   The solution to the original equation is: $y = y_p + y_c$

   $$\boxed{\Rightarrow y = 2\sin(3x) - 2\cos(3x) + c_1 e^{-3x} + c_2 e^{2x}}$$

2. Solve, using the Method of Variation of Parameters:

$y'' + y = \tan(x)$

First, find the solution to the homogenous equation $y'' + y = 0$

$\Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

$\Rightarrow y_c = c_1 e^{ix} + c_2 e^{-ix} = A\cos(x) + B\sin(x)$

To find the nonhomogeneous solution, we let $y = A(x)\cos(x) + B(x)\sin(x)$

RESTRICTION #1 We require that $A(x)$ and $B(x)$ are such that $y = A(x)\cos(x) + B(x)\sin(x)$ actually IS a solution to the non homogenous equation $y'' + y = e^{2x}$

We compute the derivatives.

$\Rightarrow y' = A'(x)\cos(x) - A(x)\sin(x) + B'(x)\sin(x) + B(x)\cos(x)$

Before we compute y'', we take advantage of the opportunity to impose a second restriction.

RESTRICTION #2 $A'(x)\cos(x) + B'(x)\sin(x) = 0$

Simplifying, y', we have: $y' = -A(x)\sin(x) + B(x)\cos(x)$

$\Rightarrow y'' = -A'(x)\sin(x) - A(x)\cos(x) + B'(x)\cos(x) - B(x)\sin(x)$

Plugging into the original equation, $y'' + y = \tan(x)$, we have:

$$\begin{array}{lllllllllll} y'' & = & -A'(x)\sin(x) & - & A(x)\cos(x) & + & B'(x)\cos(x) & - & B(x)\sin(x) & & \\ y & = & & & A(x)\cos(x) & & & + & B(x)\sin(x) & & \\ \hline y'' + y & = & -A'(x)\sin(x) & & & + & B'(x)\cos(x) & & & = & \tan(x) \end{array}$$

Using this in conjunction with restriction #2, we have:

$$\begin{array}{lllllll} & -A'(x)\sin(x) & + & B'(x)\cos(x) & & = & \tan(x) \\ \tan(x) & [A'(x)\cos(x) & + & B'(x)\sin(x)] & & = & \tan(x)\,[0] \\ \hline\hline & & & B'(x)\left[\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right] & & = & \tan(x) \end{array}$$

$\Rightarrow B'(x)\left[\frac{\sin^2(x)+\cos^2(x)}{\cos(x)}\right] = \tan(x) \Rightarrow B'(x)\left[\frac{1}{\cos(x)}\right] = \tan(x) \Rightarrow B'(x) = \sin(x)$

$\Rightarrow B(x) = -\cos(x) + C_1$

Recall:

$$\begin{array}{llllllll} y'' + y & = & & -A'(x)\sin(x) & + & B'(x)\cos(x) & = & \tan(x) \\ & & -\cot(x) & [A'(x)\cos(x) & + & B'(x)\sin(x)] & = & [0]\,(-\cot(x)) \\ \hline\hline & & & -A'(x)\left[\frac{\cos^2(x)}{\sin(x)} + \sin(x)\right] & & & = & \tan(x) \end{array}$$

$\Rightarrow -A'(x)\left[\frac{\cos^2(x)+\sin^2(x)}{\sin(x)}\right] = \tan(x) \Rightarrow -A'(x)\left[\frac{1}{\sin(x)}\right] = \tan(x) \Rightarrow A'(x) = -\frac{\sin^2(x)}{\cos(x)}$

$\Rightarrow A(x) = -\int \frac{\sin^2(x)}{\cos(x)}dx = -\int \frac{1-\cos^2(x)}{\cos(x)}dx = -\int \left(\frac{1}{\cos(x)} - \frac{\cos^2(x)}{\cos(x)}\right)dx$

$= -\int (\sec(x) - \cos(x))\,dx = -\ln|\sec(x) + \tan(x)| + \sin(x) + C_2$

The solution to the original equation, $y'' + y = \tan(x)$ is

$y = A(x)\cos(x) + B(x)\sin(x)$

$\Rightarrow y = (-\ln|\sec(x) + \tan(x)| + \sin(x) + C_2)\cos(x) + (-\cos(x) + C_1)\sin(x)$

$\Rightarrow y = -\ln|\sec(x) + \tan(x)|\cos(x) + \sin(x)\cos(x) + C_2\cos(x) - \cos(x)\sin(x) + C_1\sin(x)$

$$\boxed{\text{i.e., } y = -\ln|\sec(x) + \tan(x)|\cos(x) + C_2\cos(x) + C_1\sin(x)}$$

3. The water in a jacuzzi is heated to 100° F and then the heating elements are turned off. After 1 hour, the temperature of the water is 80° F. If the temperature of the room is a constant 70° F, what will the temperature of the water in the jacuzzi be after 3 hours?

   Let T be the temperature of the water at time $t \geq 0$. Newtons law of heating/cooling tells us that the rate ($\frac{dT}{dt}$) at which the water heats up or cools down is proportional to the difference between the temperature of the water and the temperature of the surrounding environment (room temperature), T_r.

   i.e., $\frac{dT}{dt} = k\left(T - T_r\right)$, where k is the constant of proportionality.

   Separating the variables, we have:

   $\frac{1}{(T-T_r)}dT = kdt$

   $\int \frac{1}{(T-T_r)}dT = \int kdt$

   $\Rightarrow \ln|T - T_r| = kt + C$

   $\Rightarrow \ln(T - T_r) = kt + C$ (no absolute value bars needed, since $T - T_r > 0$.)

   $\Rightarrow e^{\ln(T-T_r)} = e^{kt+C}$

   $\Rightarrow T - T_r = Ce^{kt}$

   $\Rightarrow T = T_r + Ce^{kt}$

   $\Rightarrow T = 70^\circ + Ce^{kt}$ (Room temperature is 70°)

   Recall: At time $t = 0$ hr, $T = 100^\circ$

   $\Rightarrow 100^\circ = 70^\circ + Ce^{k(0 \text{ hr})}$

   $\Rightarrow 100^\circ = 70^\circ + C$

   $\Rightarrow C = 100^\circ - 70^\circ = 30^\circ$

   Hence, $T = 70^\circ + 30^\circ e^{kt}$

   Recall Also: At time $t = 1$ hr, $T = 80^\circ$

   $\Rightarrow 80^\circ = 70^\circ + 30^\circ e^{k(1 \text{ hr})}$

   $\Rightarrow 30^\circ e^{k(1 \text{ hr})} = 80^\circ - 70^\circ$

   $\Rightarrow 30^\circ e^{k(1 \text{ hr})} = 10^\circ$

   $\Rightarrow e^{k(1 \text{ hr})} = \frac{1}{3}$

   $\Rightarrow \ln\left(e^{k(1 \text{ hr})}\right) = \ln\left(\frac{1}{3}\right)$

   $\Rightarrow k\,(1 \text{ hr}) = \ln\left(\frac{1}{3}\right) = -1.0986$

   $\Rightarrow k = \frac{-1.0986}{\text{hr}}$

   $\Rightarrow T = 70^\circ + 30^\circ e^{\frac{-1.0986}{\text{hr}}t}$

The temperature after 3 hours is given by:

$T = 70° + 30° e^{\frac{-1.0986}{\text{hr}}(3\text{ hr})} = 71.111°$

$$\boxed{T\,(3\text{ hr}) = 71.111°}$$

4. 100 kg of a radioactive substance is stored safely away. In 10 years, only 95 kg of the substance will remain. How much of the substance will remain in 100 years?

Recall: The rate $\left(\frac{dA}{dt}\right)$ at which a radioactive substance decays is proportional to the amount A of the substance present.

i.e., $\frac{dA}{dt} = kA$

Separating the variables, we have:

$\frac{1}{A}dA = kdt$

$\Rightarrow \int \frac{1}{A}dA = \int kdt$

$\Rightarrow \ln|A| = kt + C$

$\Rightarrow \ln(A) = kt + C$ (No absolute value bars needed since $A > 0$)

$\Rightarrow e^{\ln(A)} = e^{kt+C}$

$\Rightarrow A = Ce^{kt}$

Recall: 100 kg of the substance is present at time $t = 0$ years.

Then $100 \text{ kg} = A(0 \text{ years}) = Ce^{k(0 \text{ years})}$

$\Rightarrow 100 \text{ kg} = C$

$\Rightarrow A = 100 \text{ kg } e^{kt}$

Recall Also: At time $t = 10$ years, $A = 95$ kg

i.e., $95 \text{ kg} = A(10 \text{ years}) = 100 \text{ kg } e^{k(10 \text{ years})}$

$\Rightarrow 95 \text{ kg} = 100 \text{ kg } e^{k(10 \text{ years})}$

$\Rightarrow 0.95 = e^{k(10 \text{ years})}$

$\Rightarrow \ln(0.95) = \ln\left(e^{k(10 \text{ years})}\right)$

$\Rightarrow \ln(0.95) = k(10 \text{ years})$

$\Rightarrow k = \frac{\ln(0.95)}{10 \text{ years}} = \frac{-0.03567}{\text{years}} \frac{\ln(0.95)}{10} = -0.005129$

$\Rightarrow A = 100 \text{ kg } e^{-\frac{0.005129}{\text{years}}t}$

We want: A when $t = 100$ years

$\Rightarrow A = 100 \text{ kg } e^{-\frac{0.005129}{\text{years}}(100 \text{ years})}$

$\Rightarrow A = 100 \text{ kg } e^{-0.5129} = 59.876 \text{ kg}$

i.e., $\Rightarrow A = 59.876$ kg

i.e., When $t = 100$ years, $A = 59.876$ kg