MTH 3311 Test #2 - Solutions

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Instructions: Do any two of the exercises below for credit

1. Water at temperature 10° C takes 15 minutes to warm up to 20° C in a room at temperature 35° C

   (a) Find the temperature after 20 minutes

   (b) When will the temperature be 30° C?

   Given a room at constant temperature T_r and a liquid at temperature T, the rate at which the temperature warms or cools $\frac{dT}{dt}$ is proportional to the difference in the room temperature T_r and the temperature of the liquid T. (Here, t represents time)

   i.e., $\frac{dT}{dt} \propto (T_r - T)$

   or $\frac{dT}{dt} = k(T_r - T)$, where k is the constant of proportionality.

   Since this is a differential equation in the variable T, we can solve it for T. We can do this in at least two different ways.

First Way (Separate the Variables)

$\frac{dT}{dt} = k(T_r - T)$

$\Rightarrow \frac{1}{(T_r - T)} dT = kdt$

$\Rightarrow \int \frac{1}{(T_r - T)} dT = \int kdt$

$$\boxed{\begin{array}{lcl} u & = & T_r - T \\ \frac{du}{dT} & = & -1 \\ du & = & -dT \\ -du & = & dT \end{array}}$$

$$\int \underbrace{\frac{1}{(T_r - T)}}_{\frac{1}{u}} \underbrace{dT}_{-du} = \int \frac{1}{u}(-du) = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|T_r - T| + C$$

$$= -\ln(T_r - T) + C_1 \text{ (no absolute bars needed, since } T_r > T\text{)}$$

Also: $\int k dt = kt + C_2$

Thus, $\int \frac{1}{(T_r - T)} dT = \int k dt \Rightarrow -\ln(T_r - T) + C_1 = kt + C_2$

or, $\ln(T_r - T) + C_1 = -kt + C_2$

$\Rightarrow \ln(T_r - T) = -kt + C_3$

$\Rightarrow e^{\ln(T_r - T)} = e^{-kt + C_3} = C_4 e^{-kt}$

i.e., $T_r - T = C_4 e^{-kt}$

$\Rightarrow -T = -T_r + C_4 e^{-kt}$

$T = T_r + Ce^{-kt}$

Second Way (Use an Integrating Factor)

$\frac{dT}{dt} = k(T_r - T)$

$\Rightarrow \frac{dT}{dt} = kT_r - kT$

$\Rightarrow \frac{dT}{dt} + \underbrace{k}_{P(t)} T = kT_r$

Our integrating factor is $e^{\int P(t)dt} = e^{\int k dt} = e^{kt}$

$\Rightarrow e^{kt} \frac{dT}{dt} + ke^{kt}T = kT_r e^{kt}$

$\Rightarrow \frac{d}{dt}\left[e^{kt}T\right] = kT_r e^{kt}$

$\Rightarrow \frac{d}{dt}\left[e^{kt}T\right] = kT_r e^{kt}$

$\Rightarrow \int \frac{d}{dt}\left[e^{kt}T\right] dt = kT_r \int e^{kt} dt$

$\Rightarrow e^{kt}T = kT_r\left[\frac{1}{k}e^{kt}\right] + C$

$\Rightarrow e^{kt}T = T_r e^{kt} + C$

$\Rightarrow T = T_r + Ce^{-kt}$

Next (Solve for the constants C and k)

Recall: When $t = 0$ minutes, $T = 10°$

$\Rightarrow 10° = 35° + Ce^{-k(0 \text{ minutes})} = 35° + C$

i.e., $10° = 35° + C$

i.e., $-25° = C$

Thus, $T = T_r - 25° e^{-kt}$

i.e., $T = 35° - 25° e^{-kt}$

Also Recall: When $t = 15$ minutes, $T = 20°$

$\Rightarrow 20° = 35° - 25° e^{-k(15 \text{ minutes})}$

$\Rightarrow -15° = -25° e^{-k(15 \text{ minutes})}$

$\Rightarrow \frac{-15°}{-25°} = e^{-k(15 \text{ minutes})}$

$\Rightarrow 0.6 = e^{-k(15 \text{ minutes})}$

$\Rightarrow \ln(0.6) = \ln\left(e^{-k(15 \text{ minutes})}\right) = -k\,(15 \text{ minutes})$

i.e., $-0.51083 = -k\,(15 \text{ minutes})$

i.e., $0.51083 = k\,(15 \text{ minutes})$

$\Rightarrow \frac{0.51083}{15 \text{ minutes}} = k$

Thus $T = 35° - 25° e^{-\frac{0.51083}{15 \text{ minutes}} t}$

$$\boxed{\Rightarrow T = 35° - 25° e^{-\frac{0.51083}{15 \text{ minutes}} t}}$$

(a) Find the temperature after 20 minutes

$T\,(20 \text{ minutes}) = 35° - 25° e^{-\frac{0.51083}{15 \text{ minutes}}(20 \text{ minutes})} = 22.349°$

$$\boxed{\text{i.e., } T\,(20 \text{ minutes}) = 22.349°}$$

(b) When will the temperature be 30° C?

$T = 35° - 25° e^{-\frac{0.51083}{15 \text{ minutes}} t}$

$$\Rightarrow 30^\circ = 35^\circ - 25^\circ e^{-\frac{0.51083}{15\text{ minutes}}t}$$

$$\Rightarrow -5^\circ = -25^\circ e^{-\frac{0.51083}{15\text{ minutes}}t}$$

$$\Rightarrow \frac{-5^\circ}{-25^\circ} = e^{-\frac{0.51083}{15\text{ minutes}}t}$$

$$\Rightarrow 0.2 = e^{-\frac{0.51083}{15\text{ minutes}}t}$$

$$\Rightarrow \ln(0.2) = \ln\left(e^{-\frac{0.51083}{15\text{ minutes}}t}\right) = -\frac{0.51083}{15\text{ minutes}}t$$

$$\Rightarrow \frac{\ln(0.2)}{\left(-\frac{0.51083}{15\text{ minutes}}\right)} = t$$

$$\Rightarrow t = \frac{\ln(0.2)}{\left(-\frac{0.51083}{15}\right)} = 47.26\text{ minutes}$$

$$\boxed{\text{i.e., } t = 47.26\text{ minutes}}$$

2. A paratrooper and parachute weigh 240 lb. At the instant the parachute opens, he is traveling vertically downward at 40 $\frac{\text{ft}}{\text{sec}}$. If the air resistance varies directly as the instantaneous velocity, and the air resistance is 80 lb when the velocity is 20 $\frac{\text{ft}}{\text{sec}}$:

   (a) Determine the velocity at any time t.

   (b) Find the limiting velocity

First, we establish our conventions regarding direction:

$$\text{Positive Direction } \uparrow$$

We will start our stopwatch at the instant that the parachute opens.

Thus, $v\,(0 \text{ sec}) = -40\ \frac{\text{ft}}{\text{sec}}$ (The velocity is negative because the motion is in the downward direction.)

Let R be the resistance due to air.

When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 80$ lb (Because "For a velocity of $20\frac{\text{ft}}{\text{sec}}$ (in the *negative* direction), ... air resis-

tance is 80 lb.")

Also: "The force of air resistance is proportional to the velocity" i.e. $R \propto v$

$\Rightarrow R = kv$, where k is the **constant of proportionality.**

For "future reference," we will find the constant of proportionality right now.

Recall: When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 80$ lb

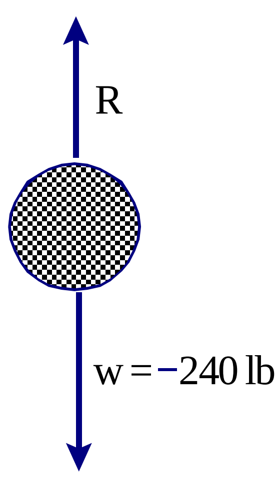
Also: $R = kv$

$\Rightarrow R = 80 \text{ lb} = k\left(-20\frac{\text{ft}}{\text{sec}}\right)$

$\Rightarrow k = \frac{80 \text{ lb}}{-20\frac{\text{ft}}{\text{sec}}} = -4\frac{\text{lb sec}}{\text{ft}}$

$\Rightarrow k = -4\frac{\text{lb sec}}{\text{ft}}$ (For "future reference")

Next: Since there is more than one force acting on the object, let's draw a force diagram of the object.



From the force diagram, the total force $F = w + R$,

where w is the **weight** of the object and R is the force on the object, due to air resistance.

Remark: To allow ourselves to model this relationship as a differential equation, we will employ a **standard trick:**

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Note well: When more than one force is acting on a free falling object, our approach will usually be to set F (the sum or all forces on the object) equal to ma.

$$\underbrace{(\text{sum of all forces})}_{F} = \underbrace{ma}_{F}$$

This is a standard approach for velocity exercises!!!

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Our "Standard Trick" yields the equation $\underbrace{w + R}_{\substack{\text{Sum of all}\\ \text{forces}}} = \underbrace{ma}_{F}$ (Eq. 1)

Recall: acceleration is the derivative of velocity. i.e., $a = \frac{dv}{dt}$

Thus, Eq. 1 can be rendered:

$$\underbrace{w + kv}_{w+R} = \underbrace{m\frac{dv}{dt}}_{ma}$$

This is a differential equation in v.

Let's solve it!

$-m\frac{dv}{dt} + kv = -w$

$$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \underbrace{\frac{w}{m}}_{Q(t)}$$

Compute the integrating factor, $e^{\int P(t)dt} = e^{\int \left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiplying both sides by the integrating factor, we have:

$e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \int \left(\frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]\right)dt = \int \frac{w}{m}e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{w}{m}\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t} = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

i.e. $e^{-\frac{k}{m}t}v = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

$\Rightarrow v = -\frac{w}{k} + e^{\frac{k}{m}t}C$

Now, let's find the constant C

Recall: $v\left(0 \text{ sec}\right) = -40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow -40\frac{\text{ft}}{\text{sec}} = v\left(0 \text{ sec}\right) = -\frac{w}{k} + e^{\frac{k}{m}(0 \text{ sec})}C = -\frac{w}{k} + C$

i.e. $-40\frac{\text{ft}}{\text{sec}} = -\frac{w}{k} + C$

$\Rightarrow C = \frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow v = -\frac{w}{k} + \left(\frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}\right) e^{\frac{k}{m}t}$

To find $\frac{w}{k}$, recall two things:

First, $k = -4\frac{\text{lb sec}}{\text{ft}}$

Next, the weight, $w = -240$ lb.

Thus, $\frac{w}{k} = \frac{-240 \text{ lb}}{-4\frac{\text{lb sec}}{\text{ft}}} = 60\ \frac{\text{ft}}{\text{sec}} *$

i.e., $\frac{w}{k} = 60\ \frac{\text{ft}}{\text{sec}}$

Finally, we want to find $\frac{k}{m}$.

Note that $w = mg$, where g is the acceleration due to gravity.

$\Rightarrow m = \frac{w}{g} = \frac{-240 \text{ lb}}{-32\frac{\text{ft}}{\text{sec}^2}} = 7.5\frac{\text{lb sec}^2}{\text{ft}}$

i.e., $m = 7.5\frac{\text{lb sec}^2}{\text{ft}}$

Hence, observe that $\frac{k}{m} = \frac{-4\frac{\text{lb sec}}{\text{ft}}}{7.5\frac{\text{lb sec}^2}{\text{ft}}} = -\frac{0.533}{\text{sec}}$

Therefore, **velocity** is given by: $v\left(t\right) = -60\frac{\text{ft}}{\text{sec}} + \left(60\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}}\right) e^{-\frac{0.533}{\text{sec}}t} = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}$

b) Therefore, **velocity** is given by: $v\left(t\right) = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}$

a) Find the limiting velocity

limiting velocity $= \lim_{t\to\infty} v\left(t\right) = \lim_{t\to\infty}\left(v\left(t\right) = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}\right) = -60\frac{\text{ft}}{\text{sec}}$

The **limiting velocity** is given by: $v\left(t\right) = -60\frac{\text{ft}}{\text{sec}}$

Ouch! At THAT speed, it's really gonna hurt when he hits the ground!!!

To find the **position** at time t, recall that the vertical position $s = \int v\left(t\right) dt$

$s\left(t\right) = \int v\left(t\right) dt = \int \left(-60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}\right) dt = -60\frac{\text{ft}}{\text{sec}}t + 20\frac{\text{ft}}{\text{sec}}\left(-\frac{1}{0.533}\text{ sec}\right) e^{-\frac{0.533}{\text{sec}}t} + C$

$\qquad = -60\frac{\text{ft}}{\text{sec}}t - \frac{20}{0.533}\text{ ft } e^{-\frac{0.533}{\text{sec}}t} + C$

i.e., $s\left(t\right)=-60\frac{\text{ft}}{\text{sec}}t-\frac{20}{0.533}\text{ ft }e^{-\frac{0.533}{\text{sec}}t}+C$

No "initial position" is given in the problem, so we will assume that the initial position is 0 ft

Thus, $0\text{ ft}=s\left(0\text{ sec}\right)=-60\frac{\text{ft}}{\text{sec}}\left(0\text{ sec}\right)-\frac{20}{0.533}\text{ ft }e^{-\frac{0.533}{\text{sec}}\left(0\text{ sec}\right)}+C$

i.e., $0\text{ ft}=-37.523\text{ ft }+C$

$\Rightarrow C=37.523\text{ ft}$

$$\boxed{\text{Thus, the vertical position is given by: } s\left(t\right)=-60\frac{\text{ft}}{\text{sec}}t-\frac{20}{0.533}fte^{-\frac{0.533}{\text{sec}}t}+37.523\text{ ft}}$$

3. The demand and supply of a certain commodity are given in terms of thousands of units, respectively, by:

$$D = 50 + 12p(t) + 2p'(t); \quad S = 450 - 8p(t) - 2p'(t).$$

At $t = 0$, the price of the commodity is 40 monetary units.

(a) Find the price at any later time and obtain its graph.

(b) Determine whether there is price stability. If there is, determine the equilibrium price.

Equating supply and demand, we have:

$50 + 12p(t) + 2p'(t) = 450 - 8p(t) - 2p'(t)$

$\Rightarrow 4p'(t) + 20p(t) = 400$

$\Rightarrow p'(t) + \underbrace{5}_{P(t)} p(t) = \underbrace{100}_{Q(t)}$

Our integrating factor is $e^{\int P(t)dt} = e^{\int 5dt} = e^{5t}$

Multiplying both sides by the integrating factor, e^{5t}, we have:

$e^{5t}p'(t) + 5e^{5t}p(t) = 100e^{5t}$

$\Rightarrow \frac{d}{dt}\left[e^{5t}p(t)\right] = 100e^{5t}$ Integrating, we have:

$\Rightarrow \int \left(\frac{d}{dt}\left[e^{5t}p(t)\right]\right) dt = \int 100e^{5t}dt$

$\Rightarrow e^{5t}p(t) = 100\left(\frac{1}{5}\right)e^{5t} + C = 20e^{5t} + C$

i.e., $e^{5t}p(t) = 20e^{5t} + C$

$\Rightarrow p(t) = 20 + e^{-5t}C$

To find the constant C, we use our initial condition $p(0) = 40$ (Because "At $t = 0$, the price of the commodity is 40 units.")

$\Rightarrow 40 = p(0) = 20 + e^{-5(0)}C = 20 + C$

i.e., $40 = 20 + C$

i.e., $20 = C$

Hence, $p(t) = 20 + 20e^{-5t}$ is the price at any time t.

To graph the function, let's consider the derivative.

$p'(t) = -100e^{-5t}$

Note that $p'(t) < 0$ for all values of t, since $e^{\text{ham sandwich}}$ is always positive.

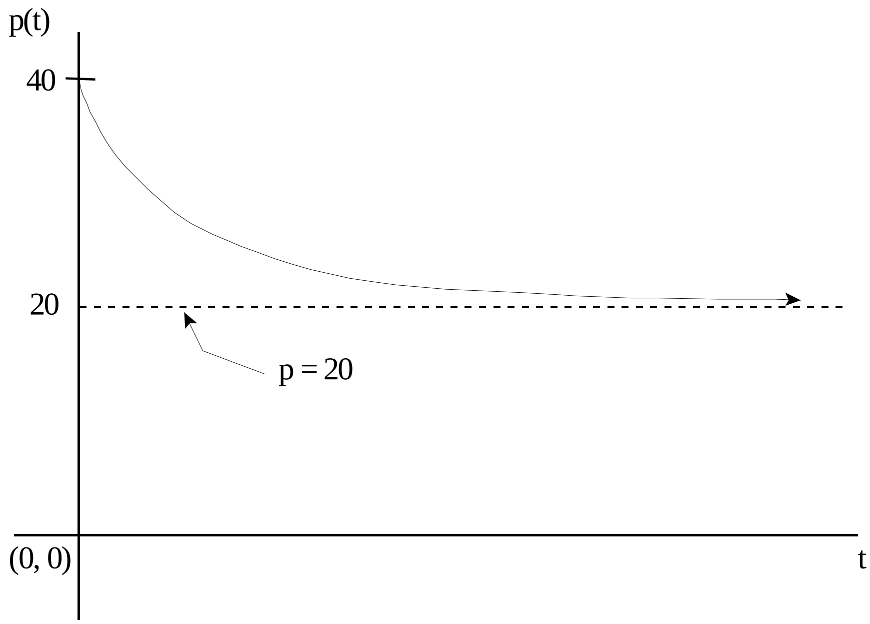
Hence, the graph of $p(t)$ is decreasing.

Next, let's consider the graph of $p(t)$ as $t \to \infty$.

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} (20 + 20e^{-5t}) = 20 + 0 = 20$

i.e., $\lim_{t\to\infty} p(t) = 20$

The graph of $y = p(t)$ is given below:



The market is **stable.** The equilibrium price is 20 units. The price will continue to decrease toward the equilibrium price $p_e = 20$.