Integration By Parts - Solutions

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11. $\int xe^{-x}dx = -xe^{-x} - e^{-x} + C$

Let u be the portion of the integrand whose derivative is simpler than itself

$u = x$

$\frac{du}{dx} = 1$

Let dv be the most complex part of the integrand that you can integrate

$dv = e^{-x}$

$u = x$	$dv = e^{-x}dx$
$\frac{du}{dx} = 1$	$v = \int dv = \int e^{-x}dx$
$du = dx$	$v = -e^{-x}$

$\int xe^{-x}dx = \int udv = uv - \int vdu = x\left(-e^{-x}\right) - \int \left(-e^{-x}\right)dx = -xe^{-x} - e^{-x} + C$

$$\boxed{\int xe^{-x}dx = -xe^{-x} - e^{-x} + C}$$

12. $\int x\cos(5x)\,dx = \frac{1}{25}\cos(5x) + \frac{1}{5}x\sin(5x) + C$

Let u be the portion of the integrand whose derivative is simpler than itself

$u = x$

$\frac{du}{dx} = 1$

Let dv be the most complex of the integrand that you can integrate

$dv = \cos(5x)\,dx$

$u = x$	$dv = \cos(5x)\,dx$
$\frac{du}{dx} = 1$	$v = \int dv = \int \cos(5x)\,dx$
$du = dx$	$v = \frac{1}{5}\sin(5x)$

$$\begin{aligned}\int x\cos(5x)\,dx &= \int u\,dv = uv - \int v\,du = x\left(\tfrac{1}{5}\sin(5x)\right) - \int \left(\tfrac{1}{5}\sin(5x)\right)dx \\ &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\int \sin(5x)\,dx = \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\left(-\tfrac{1}{5}\cos(5x)\right) + C \\ &= \tfrac{1}{5}x\sin(5x) + \tfrac{1}{25}\cos(5x) + C\end{aligned}$$

$$\boxed{\int x\cos(5x)\,dx = \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C}$$

13. $\int x^2 \ln(x)\, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

Let u be a transcendental function whose derivative is algebraic

(The functions that fit this description are $\ln(x)$ and inverse trig functions)

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

Let dv be everything else

$dv = x^2 dx$

$u = \ln(x)$	$dv = x^2 dx$
$\frac{du}{dx} = \frac{1}{x}$	$v = \int dv = \int x^2 dx$
$du = \frac{1}{x} dx$	$v = \frac{1}{3}x^3$

$$\int x^2 \ln(x)\, dx = \int \ln(x)\, x^2 dx = \int u dv = uv - \int v du = \ln(x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right) \frac{1}{x} dx$$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\frac{x^3}{3} + C$$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$$

$$\boxed{\int x^2 \ln(x)\, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

14. $\int x\tan^{-1}(x)\,dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C$

Let u be a transcendental function whose derivative is algebraic

(The functions that fit this description are $\ln(x)$ and inverse trig functions)

$u = \tan^{-1}(x)$

$\frac{du}{dx} = \frac{1}{1+x^2}$

Let dv be everything else

$dv = xdx$

$u = \tan^{-1}(x)$	$dv = xdx$
$\frac{du}{dx} = \frac{1}{x^2+1}$	$v = \int dv = \int xdx$
$du = \frac{1}{x^2+1}dx$	$v = \frac{1}{2}x^2$

$$\int x\tan^{-1}(x)\,dx = \int \tan^{-1}(x)\,x\,dx = \int udv = uv - \int vdu = \tan^{-1}(x)\left(\frac{1}{2}x^2\right) - \int\left(\frac{1}{2}x^2\right)\frac{1}{x^2+1}dx$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int\frac{x^2}{x^2+1}dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int\left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\left(x - \tan^{-1}(x)\right) + C$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C$$

$$\boxed{\int x\tan^{-1}(x)\,dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C}$$

15. $\int e^x \sin(x)\, dx = -\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C$

When integrating the product of two functions whose "derivatives are cyclic," the choice of u and dv is arbitrary.

e^x "has derivatives that are cyclic," in the sense that if we differentiate e^x repeatedly, we get back the original function e^x.

$\sin(x)$ and $\cos(x)$ "have derivatives that are cyclic," in the sense that if we differentiate these functions repeatedly, we get back the original function, or a constant multiple of the original function.

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

etc.

Arbitrarily, we let $u = e^x$ and $dv = \sin(x)\, dx$

$u = e^x$	$dv = \sin(x)\, dx$
$\frac{du}{dx} = e^x$	$v = \int dv = \int \sin(x)\, dx$
$du = e^x dx$	$v = -\cos(x)$

$\int e^x \sin(x)\, dx = \int u dv = uv - \int v du = e^x(-\cos(x)) - \int(-\cos(x))\, e^x dx$

$= -e^x \cos(x) + \int e^x \cos(x)\, dx$

Repeat the process of Integration by Parts

(Continued . . . next page)

Rule of Thumb: When performing multiple iterations of Integration by Parts, do not switch the roles of u and dv. If u is the exponential function in the first iteration, then u should be the exponential function in the next iteration. If dv is the trig function in the first iteration, then dv should be the trig function in the next iteration.

$\int e^x \sin(x)\, dx = \ldots = -e^x \cos(x) + \int e^x \cos(x)\, dx$

$u = e^x$	$dv = \cos(x)\, dx$
$\frac{du}{dx} = e^x$	$v = \int dv = \int \cos(x)\, dx$
$du = e^x dx$	$v = \sin(x)$

$$\int e^x \sin(x)\, dx = \ldots = -e^x \cos(x) + \int e^x \cos(x)\, dx = -e^x \cos(x) + \int u dv$$
$$= -e^x \cos(x) + uv - \int v du = -e^x \cos(x) + e^x \sin(x) - \int \sin(x)\, e^x dx$$

Hey! We're right back where we started!

This is not a problem! When this happens, we solve for the integral algebraically.

$\int e^x \sin(x)\, dx = -e^x \cos(x) + e^x \sin(x) - \int \sin(x)\, e^x dx$

$\Rightarrow 2\int e^x \sin(x)\, dx = -e^x \cos(x) + e^x \sin(x)$

$\Rightarrow \int e^x \sin(x)\, dx = -\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C$

$$\boxed{\int e^x \sin(x)\, dx = -\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C}$$