MTH 1126 Test #2 - Part 2 11am Class - Solutions

Spring 2022

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute the length of the arc of the graph of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2$ from the point $(0, 2)$ to the point $(6, f(6))$.

Arclength $= \int_{x=0}^{x=6} \sqrt{(f'(x))^2 + 1}dx$

$f'(x) = 2x^{\frac{1}{2}}$

$(f'(x))^2 = \left(2x^{\frac{1}{2}}\right)^2 = 4x$

$(f'(x))^2 + 1 = 4x + 1$

Arclength $= \int_{x=0}^{x=6} \sqrt{(f'(x))^2 + 1}dx = \int_{x=0}^{x=6} \sqrt{4x+1}dx = \int_{x=0}^{x=6} \underbrace{(4x+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{1}{4}du}$

$= \int_{u=1}^{u=25} u^{\frac{1}{2}} \frac{1}{4} du = \frac{1}{4} \int_{u=1}^{u=25} u^{\frac{1}{2}} du = \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{u=1}^{u=25} = \frac{1}{6} \left[(25)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$

$= \frac{1}{6}[125 - 1] = \frac{62}{3}$

Arc Length $= \frac{62}{3}$

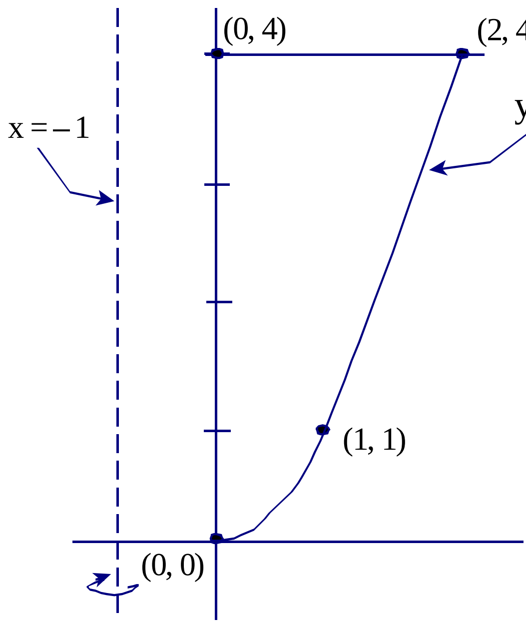
Scratchwork
$u = 4x + 1$
$\frac{du}{dx} = 4$
$du = 4dx$
When $x = 0$, $u = 4x + 1 = 1$
When $x = 6$, $u = 4x + 1 = 25$

2. Compute the volume of the solid of revolution generated by revolving the bounded region described below about the line $x = -1$. (Use the "Shell Method.")

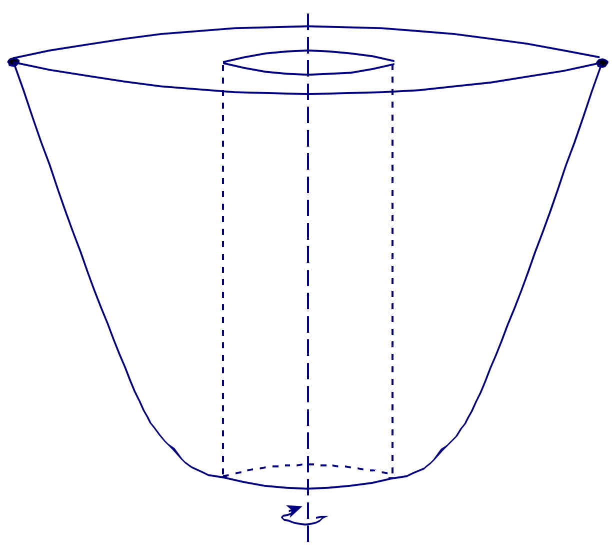
The region bounded by: the y-axis, the graph $y = x^2$, and the line $y = 4$

Use the "five step method" (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

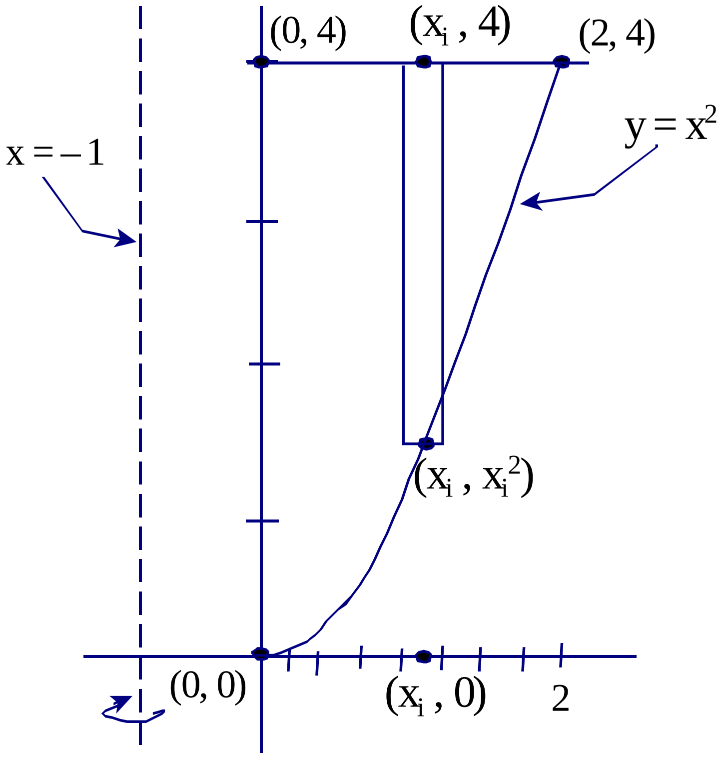
i) Graph the bounded region.



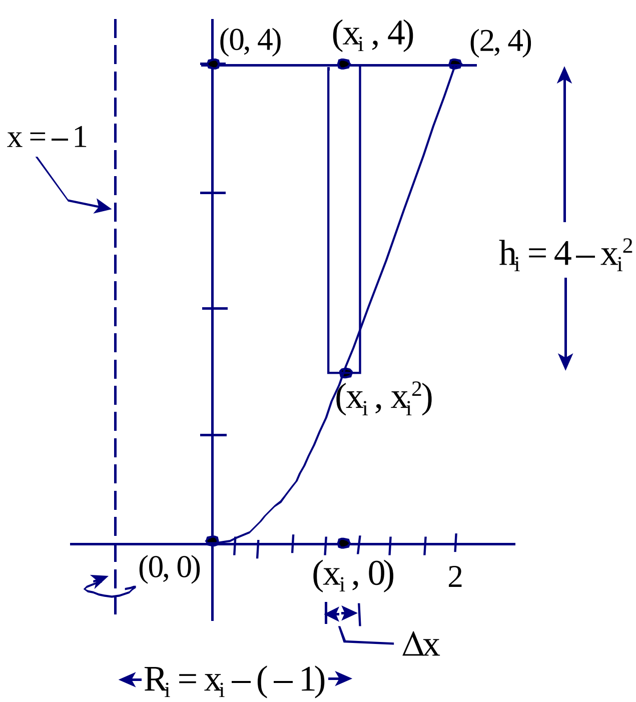
The solid of revolution looks like this:



ii) Sketch a rectangle of width Δx parallel ("shell-parallel") to the axis of revolution, and partition the interval spanned by the rectangles.



iii) Revolve the ith rectangle about the axis of revolution and compute the volume of the ith shell, Vol_i



$\text{Vol}_i = 2\pi R_i h_i \triangle x$

$= 2\pi \left(x_i - (-1)\right)\left(4 - x_i^2\right) \triangle x$

$= 2\pi \left(x_i + 1\right)\left(4 - x_i^2\right) \triangle x$

$= 2\pi \left(-x_i^3 - x_i^2 + 4x_i + 4\right) \triangle x$

iv) Approximate the volume of the solid by adding up the volumes of the shells

$Vol \approx \sum_{i=1}^{n} 2\pi \left(-x_i^3 - x_i^2 + 4x_i + 4\right) \triangle x$

v) Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(-x_i^3 - x_i^2 + 4x_i + 4\right) \triangle x = \int_0^2 2\pi \left(-x^3 - x^2 + 4x + 4\right) dx$

$= 2\pi \int_0^2 \left(-x^3 - x^2 + 4x + 4\right) dx = 2\pi \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 2x^2 + 4x\right]_0^2$

$= 2\pi \left[\left(-\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 + 2(2)^2 + 4(2)\right) - \left(-\frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 + 2(0)^2 + 4(0)\right)\right]$

$= \frac{56\pi}{3}$

$\boxed{\text{Volume} = \frac{56\pi}{3}}$

Extra Wow! (5 points!)

Compute the length of the arc of the graph of the function $f(x) = x^4 + \frac{1}{32}x^{-2}$ from the point $\left(1, \frac{33}{32}\right)$ to the point $\left(2, \frac{2049}{128}\right)$.

Arclength $= \int_{x=1}^{x=2} \sqrt{(f'(x))^2 + 1} dx$

$f'(x) = 4x^3 - \frac{1}{16}x^{-3}$

$(f'(x))^2 = \left(4x^3 - \frac{1}{16}x^{-3}\right)^2 = 16x^6 - \frac{1}{2} + \frac{1}{256}x^{-6}$

Arclength $= \int_{x=1}^{x=2} \sqrt{(f'(x))^2 + 1} dx = \int_{x=1}^{x=2} \sqrt{\left(16x^6 - \frac{1}{2} + \frac{1}{256}x^{-6}\right) + 1} dx$

$$= \int_{x=1}^{x=2} \sqrt{16x^6 + \frac{1}{2} + \frac{1}{256}x^{-6}} dx = \int_{x=1}^{x=2} \sqrt{(4x^3)^2 + \frac{1}{2} + \left(\frac{1}{16}x^{-3}\right)^2} dx$$

$$= \int_{x=1}^{x=2} \sqrt{\left(4x^3 + \frac{1}{16}x^{-3}\right)^2} dx = \int_{x=1}^{x=2} 4x^3 + \frac{1}{16}x^{-3} dx = \left[x^4 - \frac{1}{32}x^{-2}\right]_{x=1}^{x=2}$$

$$= \left[\left((2)^4 - \frac{1}{32}(2)^{-2}\right) - \left((1)^4 - \frac{1}{32}(1)^{-2}\right)\right] =$$

Arc Length $= \frac{1923}{128}$