MTH 1125 (2 pm - Pod A) Test #3 - Solutions

Fall 2020

Pat Rossi — Name ________________________

Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 + 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 6x^2 + 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 6x^2 + 6x = 0$

   $\Rightarrow 6x(x+1) = 0$

   $\Rightarrow 6x = 0$ or $(x+1) = 0$

   $\Rightarrow x = 0$ and $x = -1$ are critical numbers.

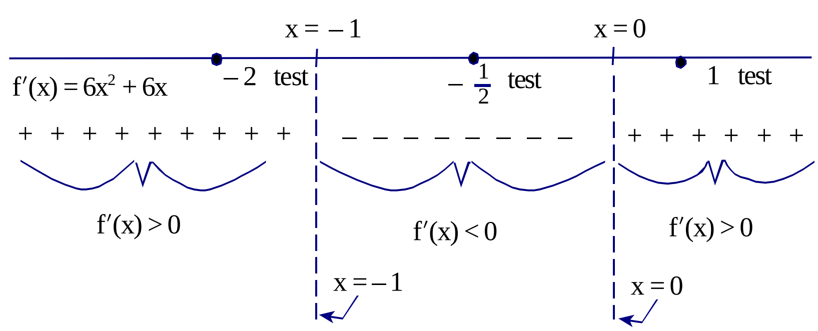
   b. "Type b" ($f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

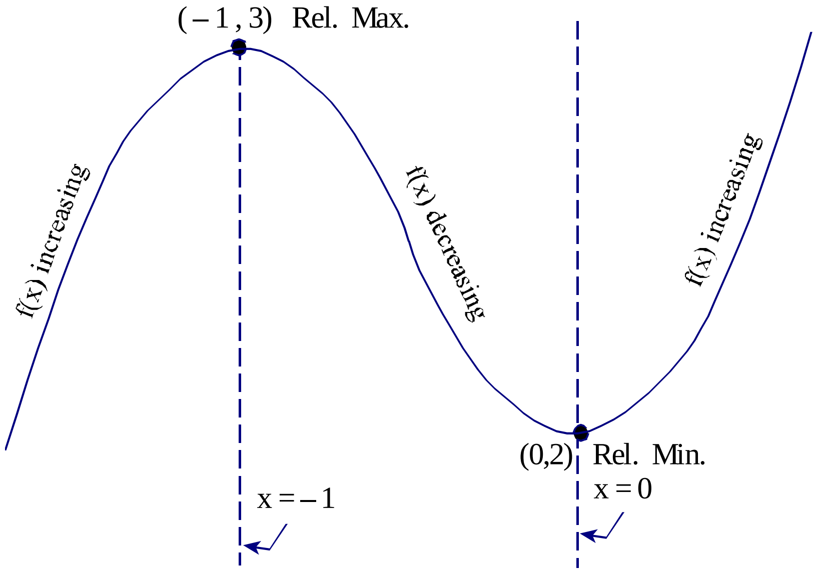
3. Pick a "test point" from each interval to plug into $f'(x)$



$f(x)$ is **increasing** on the interval(s) $(-\infty, -1)$ and $(0, \infty)$
(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-1, 0)$
(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-1, f(-1)) = (-1, 3)$

Rel Min $(0, f(0)) = (0, 2)$

2. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{15}{2}x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = x^3 + 6x^2 - 15x + 6$

      $f''(x) = 3x^2 + 12x - 15$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 3x^2 + 12x - 15 = 0$

         $\Rightarrow 3(x^2 + 4x - 5) = 0$

         $\Rightarrow 3(x+5)(x-1) = 0$

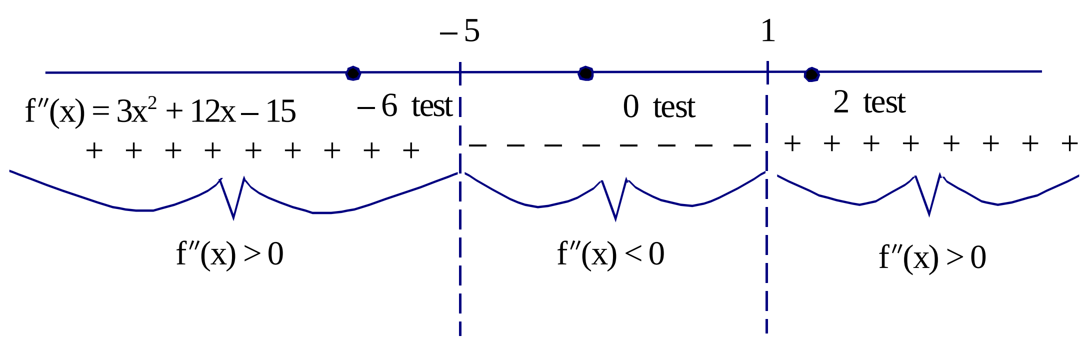
         $x = -5, 1$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined)

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -5)$ and $(1, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-5, 1)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -5$ and $x = 1$, the points:
$(-5, f(-5)) = \left(-5, -\frac{1233}{4}\right)$
and
$(1, f(1)) = \left(1, \frac{15}{4}\right)$ **are** points of inflection.

3. $f(x) = x^3 + 6x^2 - 15x + 3$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 3x^2 + 12x - 15$

a. "Type a" ($f'(x) = 0$)

$f'(x) = 3x^2 + 12x - 15 = 0$

$\Rightarrow 3(x^2 + 4x - 5) = 0$

$\Rightarrow (x + 5)(x - 1) = 0$

$\Rightarrow x = -5, 1$ are "type a" critical numbers

Since $x = -5$ is not in the interval $[-2, 2]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = (-2)^3 + 6(-2)^2 - 15(-2) + 3 = 49$ $\leftarrow$ Abs Max Value

$f(1) = (1)^3 + 6(1)^2 - 15(1) + 3 = -5$ $\leftarrow$ Abs Min Value

$f(2) = (2)^3 + 6(2)^2 - 15(2) + 3 = 5$

The Abs Max Value is 49 (attained at $x = -2$)

The Abs Min Value is -5 (attained at $x = 1$)

4. $f(x) = x^{\frac{20}{7}} - \frac{10}{3}x^{\frac{6}{7}} + 4$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

   $f'(x) = \frac{20}{7}x^{\frac{13}{7}} - \frac{20}{7}x^{-\frac{1}{7}} = \frac{20x^{\frac{13}{7}}}{7} - \frac{20}{7x^{\frac{1}{7}}} = \frac{20x^{\frac{13}{7}}}{7}\frac{x^{\frac{1}{7}}}{x^{\frac{1}{7}}} - \frac{20}{7x^{\frac{1}{7}}} = \frac{20x^2-20}{7x^{\frac{1}{7}}}$

   i.e., $f'(x) = \frac{20x^2-20}{7x^{\frac{1}{7}}}$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 0$ and solve for x

      $\Rightarrow f'(x) = \frac{20x^2-20}{7x^{\frac{1}{7}}} = 0$

      $\Rightarrow 20x^2 - 20 = 0$

      $\Rightarrow x^2 - 1 = 0$

      $\Rightarrow (x+1)(x-1) = 0$

      $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

   b. "Type b" $(f'(c)$ is undefined)

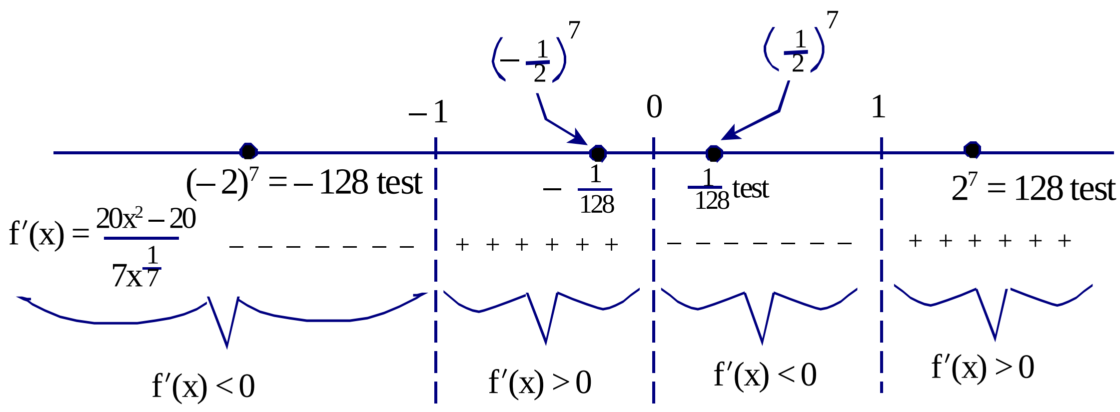
      Look for x-value that causes division by zero.

      $\Rightarrow 7x^{\frac{1}{7}} = 0$

      $\Rightarrow x = 0$ "type b" critical number

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



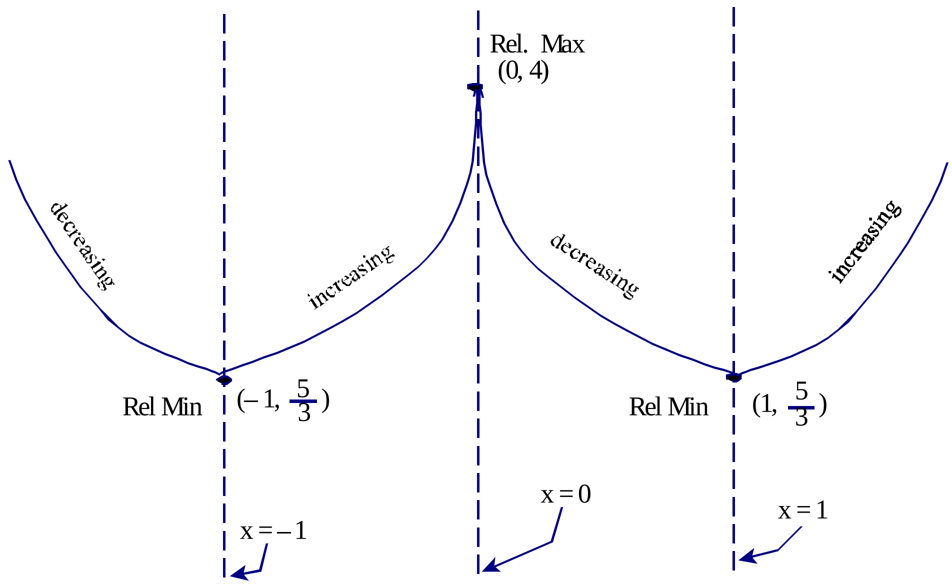
$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on those intervals)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



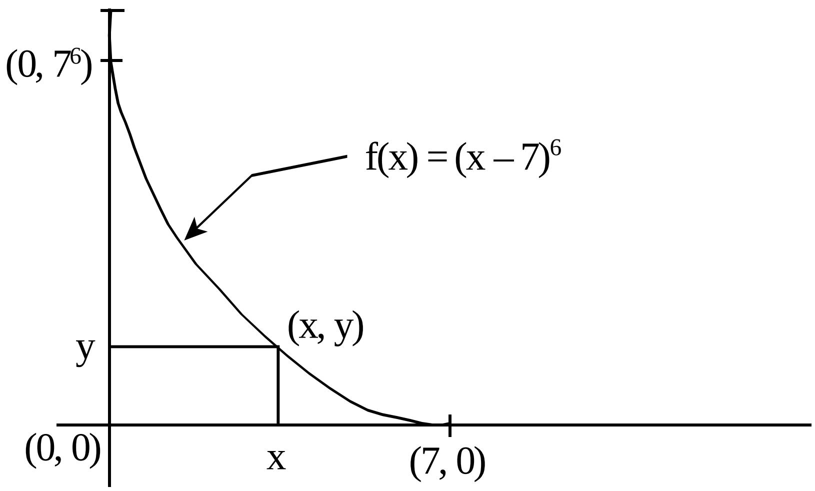
Rel Minimums: $(-1, f(-1)) = \left(-1, \frac{5}{3}\right)$

and $(1, f(1)) = \left(1, \frac{5}{3}\right)$

Rel Maximum: $(0, f(0)) = (0, 4)$

5. A rectangle is inscribed in the region bounded by the positive x-axis, the positive y-axis, and the graph of $f(x) = (x-7)^6$ as shown below. Determine the value of x that makes the area of the rectangle as large as possible.

REMARK: When you create the area function $A(x)$, **do not simplify (i.e. "multiply it out") before you compute the derivative.** If you simplify (i.e. "multiply it out") the area function $A(x)$ before you compute the derivative, it will be very difficult to find the critical numbers.



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (x-7)^6$.

   Hence, the y-coordinate of the point (x, y) is $y = (x-7)^6$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x-7)^6$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 7$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 14]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = \frac{d}{dx}\left[x(x-7)^6\right] = \underbrace{(x-7)^6 + 6x(x-7)^5}_{\text{Product Rule}}$

      i.e., $A'(x) = (x-7)^6 + 6x(x-7)^5$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = (x-7)^6 + 6x(x-7)^5 = 0$

         $\Rightarrow (x-7)^6 + 6x(x-7)^5 = 0$

         $\Rightarrow (x-7)^5\left[(x-7) + 6x\right] = 0$

         $\Rightarrow (x-7)^5(7x-7) = 0$

         $\Rightarrow x = 7$ and $x = 1$ are critical numbers

      b. "Type b" $(f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)((0)-7)^6 = 0$

      $A(1) = (1)((1)-7)^6 = 6^6 =$ a really big number $\longleftarrow$ Abs Max Value

      $A(7) = (7)((7)-7)^4 = 0$

5. Make sure that we've answered the original question.

   "Determine the value of x ..."

   $x = 1$