MTH 2215 Practice Test 3 –Version #2 - Solutions

Spring 2021

Pat Rossi Name ______________________

Show CLEARLY how you arrive at your answers.

1. Give the quotient and the remainder when:

   (a) 859 is divided by 14.

   $859 = \underbrace{(61)}_{q}\underbrace{(14)}_{b} + \underbrace{5}_{r}$

   $\boxed{\text{The quotient } q = 61 \quad \text{The remainder } r = 5}$

   (b) 731 is divided by 25.

   $731 = \underbrace{(29)}_{q}\underbrace{(25)}_{b} + \underbrace{6}_{r}$

   $\boxed{\text{The quotient } q = 29 \quad \text{The remainder } r = 6}$

   (c) -481 is divided by 32.

   $-481 = \underbrace{(-16)}_{q}\underbrace{(32)}_{b} + \underbrace{31}_{r}$

   $\boxed{\text{The quotient } q = -16 \quad \text{The remainder } r = 31}$

   Remark: Note that the following is NOT correct, because the remainder r must be non-negative.

   i.e., $0 \leq r < b$

   $-481 = \underbrace{(-15)}_{q}\underbrace{(32)}_{b} + \underbrace{(-1)}_{r}$ (Again, this is INCORRECT)

2. Evaluate the quantities:

(a) $37 \pmod{11} \equiv$

Recall: the value of $a \pmod{m}$ is the **remainder** when a is divided by m.

$37 = (3)(11) + \underbrace{4}_{r}$

$$\boxed{37 \pmod{11} \equiv 4}$$

(b) $42 \pmod{6} \equiv$

Recall: the value of $a \pmod{m}$ is the **remainder** when a is divided by m.

$42 = (7)(6) + \underbrace{0}_{r}$

$$\boxed{42 \pmod{6} \equiv 0}$$

(c) $73 \pmod{21} \equiv$

Recall: the value of $a \pmod{m}$ is the **remainder** when a is divided by m.

$73 = (3)(21) + \underbrace{10}_{r}$

$$\boxed{73 \pmod{21} \equiv 10}$$

(d) $-59 \pmod{4} \equiv$

Recall: the value of $a \pmod{m}$ is the **remainder** when a is divided by m.

$-59 = (-15)(4) + \underbrace{1}_{r}$

$$\boxed{-59 \pmod{4} \equiv 1}$$

3. Determine whether the integers below are congruent modulo the given number:

   Recall: $a \equiv b \pmod{m}$ exactly when $(a - b) = km$ for some integer k

   Alternatively: $a \equiv b \pmod{m}$ exactly when a and b have the same remainder when divided by m.

   (a) $216 \equiv 58 \pmod{5}$

   Observe: $216 = (43)(5) + 1 \qquad \text{and } 58 = (11)(5) + 3$

   (i.e., 216 and 58 do NOT have the same remainder when divided by 5.

   $\boxed{\text{Therefore, } 216 \not\equiv 58 \pmod{5}}$

   Alternatively: $216 - 58 = 158 = (31)(5) + 3$

   i.e., $216 - 58 \neq k(5) \quad \text{for } k \in \mathbb{Z}$

   $\boxed{\text{Therefore, } 216 \not\equiv 58 \pmod{5}}$

   (b) $50 \equiv 203 \pmod{17}$

   Observe: $50 = (2)(17) + 16 \qquad \text{and } 203 = (11)(17) + 16$

   (i.e., 50 and 203 have the same remainder when divided by 17.

   $\boxed{\text{Therefore, } 50 \equiv 203 \pmod{17}}$

   Alternatively: $50 - 203 = -153 = (-9)(17)$

   $\boxed{\text{Therefore, } 50 \equiv 203 \pmod{17}}$

4. Convert the decimal (base 10) representation of each integer into the equivalent hexadecimal (base 16) expansion

   To convert to base 16, apply the Division Algorithm to the number, using 16 as a divisor. Note the quotient and the remainder. Apply the Division Algorithm to the quotient, using 16 as a divisor. Repeat with each succeeding quotient.

   The digits of the base 16 representation (hexadecimal) going from Left to Right, are the **remainders – in reverse order.**

   (a) 543_{10}

$$543 = (33)(16) + 15$$

$$33 = (2)(16) + 1$$

$$2 = (0)(16) + 2$$

The Hexadecimal representation of the number (going Left to Right) is the string of remainders in reverse order (2, 1, 15 or 21F, using hexadecinal digits)

i.e. $543_{10} = 21F_{16}$

(b) 4211_{10}

$$4211 = (263)(16) + 3$$

$$263 = (16)(16) + 7$$

$$16 = (1)(16) + 0$$

$$1 = (0)(16) + 1$$

The Hexadecimal representation of the number (going Left to Right) is the string of remainders in reverse order (1, 0, 7, 3 or 1073, using hexadecinal digits)

$$\boxed{\text{i.e. } 4211_{10} = 1073_{16}}$$

5. Convert the decimal (base 10) representation of each integer into the equivalent binary (base 2) expansion

   The easiest way to do this may be to convert the base 10 representation to base 16 representation and then replace each hexadecimal digit with its binary 4-digit equivalent.

   To convert to base 16, apply the Division Algorithm to the number, using 16 as a divisor. Note the quotient and the remainder. Apply the Division Algorithm to the quotient, using 16 as a divisor. Repeat with each succeeding quotient.

   The digits of the base 16 representation (hexadecimal) going from Left to Right, are the **remainders – in reverse order.**

   (a) 97

$$97 = (6)(16) + 1$$

$$6 = (0)(16) + 6$$

The Hexadecimal representation of the number (going Left to Right) is the string of remainders in reverse order

i.e. $97_{10} = 61_{16}$

Now replace each hexadecimal digit with it's 4-digit binary equivalent.

$97_{10} = 61_{16} = 0110 \quad 0001_2$

$$\boxed{\text{i.e., } 97_{10} = (0110 \quad 0001)_2}$$

(b) 245

$$245 = (15)(16) + 5$$

$$15 = (0)(16) + 15$$

The Hexadecimal representation of the number (going from Left to Right) is the string of remainders in reverse order

i.e. $(245)_{10} = (F5)_{16}$

Now replace each hexadecimal digit with it's 4-digit binary equivalent.

$(245)_{10} = (F5)_{16} = (1111 \quad 0101)_2$

$$\boxed{\text{i.e., } (245)_{10} = (1111 \quad 0101)_2}$$

6. Convert the binary (base 2) representation of each integer into the equivalent decimal (base 10) expansion

(a) $(110011)_2 =$

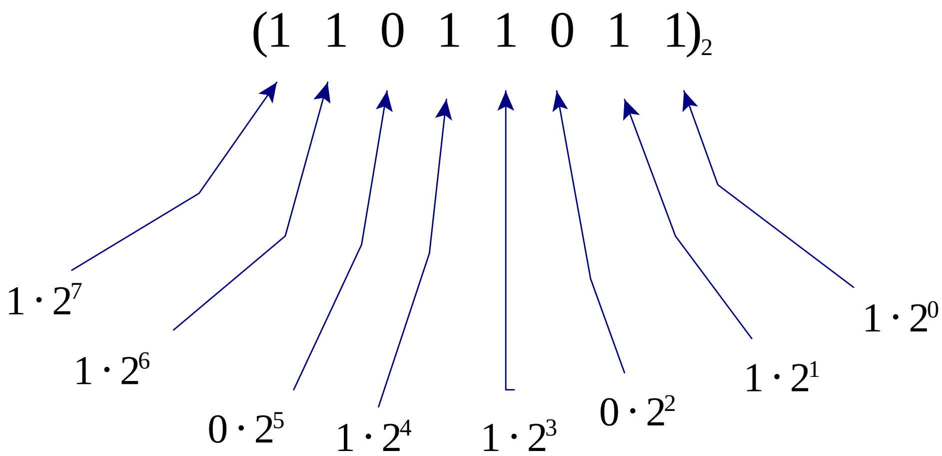
$$(1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1)_2$$

$$1 \cdot 2^5 \qquad 1 \cdot 2^4 \qquad 0 \cdot 2^3 \qquad 0 \cdot 2^2 \qquad 1 \cdot 2^1 \qquad 1 \cdot 2^0$$

$(101101)_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 51$

$$\boxed{(101101)_2 = 51_{10}}$$

(b) $(11011011)_2 =$



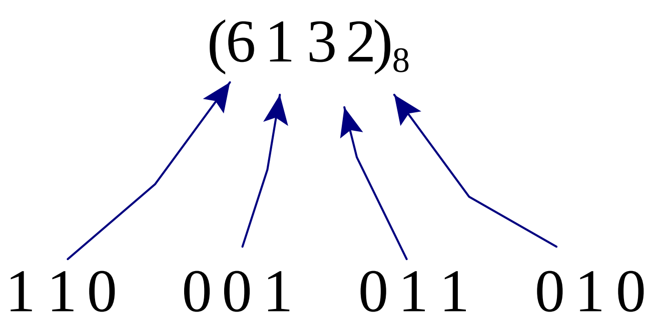
$(11011011)_2 = 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 219$

$$\boxed{(11011011)_2 = 219_{10}}$$

7. Convert the octal (base 8) representation of each integer into the equivalent binary (base 2) expansion

(a) $(6132)_8 =$

Each digit in Base 8 has a unique 3 digit binary representation. Replace each digit in Base 8 with its unique binary equivalent.



$$\boxed{(6132)_8 = 110\ \ 001\ \ 011\ \ 010}$$

(b) $(405)_8 =$

Each digit in Base 8 has a unique 3 digit binary representation. Replace each digit in Base 8 with its unique binary equivalent.

$$(4\ 0\ 5)_8$$

$$100 \quad 000 \quad 101$$

$$\boxed{(405)_8 = 100\ \ 000\ \ 101}$$

8. Convert the binary (base 2) representation of each integer into the equivalent octal (base 8) representation.

(a) $(11\ \ 100\ \ 101)_2 =$

The trick here is to group the binary digits into groups of three, adding zeros to the left of the left-most group, as needed.

Then replace each group of three binary digits with its octal equivalent

$$(\underbrace{0\ \ 1\ \ 1}_{3}\quad \underbrace{1\ \ 0\ \ 0}_{4}\quad \underbrace{1\ \ 0\ \ 1}_{5})_2$$

$$\boxed{(11\ \ 100\ \ 101)_2 = (345)_8}$$

(b) $(110\ \ 001\ \ 100)_2 =$

Group the binary digits into groups of three, adding zeros to the left of the left-most group, as needed.

Then replace each group of three binary digits with its octal equivalent

$$(\underbrace{1\ 1\ 0}_{6}\quad \underbrace{0\ 0\ 1}_{1}\quad \underbrace{1\ \ 0\ \ 0}_{4})_2$$

$$\boxed{(110\ \ 001\ \ 100)_2 = (614)_8}$$

9. Convert the hexadecimal (base 16) representation of each integer into the equivalent binary (base 2) expansion

Each digit in Base 16 has a unique 4 digit binary representation. Replace each digit in Base 16 with its unique binary equivalent.

(a) $(B71)_{16} =$

$$(\mathrm{B}\ \ 7\ \ 1)_{16}$$

$$1\,0\,1\,1\quad 0\,1\,1\,1\quad 0\,0\,0\,1$$

$$\boxed{(B71)_{16} = (1011\ \ 0111\ \ 0001)_2}$$

(b) $(C8E)_{16} =$

$$(\mathrm{C}\ \ 8\ \ \mathrm{E})_{16}$$

$$1100 \quad 1000 \quad 1110$$

$$\boxed{(C8E)_{16} = (1100\ \ 1000\ \ 1110)_2}$$

10. Convert the binary (base 2) representation of each integer into the equivalent hexadecimal (base 16) representation

    The trick here is to group the binary digits into groups of four, adding zeros to the left of the left-most group, as needed.

    Then replace each group of four binary digits with its hexadecimal equivalent

    (a) $(0111\ \ 0011\ \ 1111)_2 =$

    $$(\underbrace{0\ 1\ 1\ 1}_{7}\quad \underbrace{0\ 0\ 1\ 1}_{3}\quad \underbrace{1\ 1\ 1\ 1}_{\mathrm{F}})_2$$

    $$\boxed{(0111\ \ 0011\ \ 1111)_2 = (73F)_{16}}$$

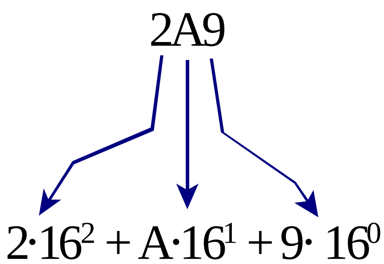
    (b) $(101\ \ 1110\ \ 0110)_2 =$

    $$(\underbrace{0\ 1\ 0\ 1}_{5}\quad \underbrace{1\ 1\ 1\ 0}_{\mathrm{E}}\quad \underbrace{0\ 1\ 1\ 0}_{6})_2$$

    $$\boxed{(101\ \ 1110\ \ 0110)_2 = (5E6)_{16}}$$

11. Convert the hexadecimal (base 16) representation of each integer into the equivalent base 10 representation

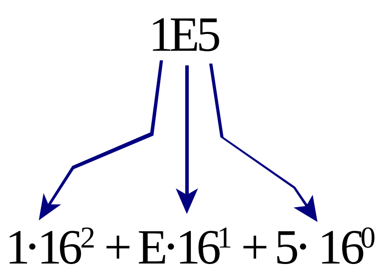
(a) $2A9_{16}$



$2A9_{16} = 2 \cdot 16^2 + A \cdot 16^1 + 9 \cdot 16^0 = 2 \cdot 16^2 + 10 \cdot 16^1 + 9 \cdot 16^0 = 681_{10}$

i.e., $2A9_{16} = 681_{10}$

(b) $1E5_{16}$



$1E5_{16} = 1 \cdot 16^2 + E \cdot 16^1 + 5 \cdot 16^0 = 1 \cdot 16^2 + 14 \cdot 16^1 + 5 \cdot 16^0 = 485_{10}$

i.e., $1E5_{16} = 485_{10}$

12. Which positive integers less than 18 are relatively prime to 18?

Any integer that is relatively prime to 18 must not have any of the same prime factors that 18 has.

$18 = 2 \cdot 3^2$, so all positive integers n with $2 \leq n \leq 18$, that do not have factors 2 and 3 are relatively prime to 18.

Therefore, 5, 7, 11, 13, and 17 are relatively prime to 18.

13. Which positive integers less than 36 are relatively prime to 36?

Any integer that is relatively prime to 36 must not have any of the same prime factors that 36 has.

$36 = 2^2 \cdot 3^2$, so all positive integers n with $2 \leq n \leq 36$, that do not have factors 2 and 3, are relatively prime to 36.

$\boxed{\text{Therefore, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35 are relatively prime to 36.}}$

14. What are the greatest common divisors of the integers $a = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2$ and $b = 2^2 \cdot 3^4 \cdot 5^2 \cdot 11^3$?

To find the $\gcd(a, b)$:

(a)
1. find the prime factors $p_1, p_2, \ldots, p_r$ that both a and b have in common
2. find the highest power n_i of each prime factor p_i that appears in both a and b
3. $\gcd(a, b) = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_r^{n_r}$

$a = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2$

$b = 2^2 \cdot 3^4 \cdot 5^2 \cdot 11^3$

$2, 3, 5$ are prime factors of both a and b

The highest powers of $2, 3,$ and 5 that appear in both a and b are $2, 2,$and 2, respectively

$\boxed{\gcd(a, b) = 2^2 \cdot 3^2 \cdot 5^2}$

15. What are the greatest common divisors of the integers 225 and 162 ?

To find the gcd $(225, 162)$:

(a) 1. find the prime factors $p_1, p_2, \ldots, p_r$ that both 225 and 162 have in common

2. find the highest power n_i of each prime factor p_i that appears in both 225 and 162

3. $\gcd(225, 162) = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_r^{n_r}$

$225 = 3 \cdot 75 = 3 \cdot 3 \cdot 25 = 3 \cdot 3 \cdot 5 \cdot 5$

i.e., $225 = 3^2 \cdot 5^2$

$162 = 2 \cdot 81 = 2 \cdot 3 \cdot 27 = 2 \cdot 3 \cdot 3 \cdot 9 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

i.e., $162 = 2^1 \cdot 3^4$

3 is the only prime factor of both 225 and 162

The highest power of 3 that appears in both 225 and 162 is 2

$$\boxed{\gcd(225, 162) = 3^2}$$